

Diploma Thesis

Classical and Quantum Dilaton Gravity in Two Dimensions with Fermions

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By

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Abstract

In this thesis the first order formulation of generalized dilaton gravities in two dimensions coupled to a Dirac fermion is considered. After a Hamiltonian analysis of the gauge symmetries and constraints of the theory and fixing Eddington-Finkelstein gauge by use of the Batalin-Vilkovisky-Fradkin method, the system is quantized in the Feynman path integral approach. It turns out that the path integral over the dilaton gravity sector can be evaluated exactly, while in the matter sector perturbative methods are applied. The gravitationally induced four-fermi scattering vertices as well as asymptotic states are calculated, and – as for dilaton gravities coupled to scalar fields – a “virtual black hole” is found to form as an intermediary geometric state in scattering processes. The results are compared to the well-known scalar case, and evidence for bosonization in this context is found.

Das ewig Unbegreifliche an der Welt ist ihre Begreiflichkeit.

Werner Karl Heisenberg

Contents

1	Introduction	1
2	Classical Analysis	5
2.1	Dilaton Gravity in Two Dimensions with Fermions	5
2.1.1	Equivalence between First and Second Order Formulation	7
2.1.2	Classical Solutions	8
2.1.3	Conformally Related Theories	10
2.1.4	Fermions	11
2.2	Hamiltonian Analysis	12
2.2.1	Primary and Secondary Constraints	13
2.2.2	Algebra of Secondary Constraints	16
2.2.3	Relation to the Conformal Algebra	18
2.3	BRST Gauge Fixing	19
3	Nonperturbative Quantization of Geometry	23
3.1	Ghosts and Second Class Constraints	23
3.2	On the Path Integral Measure	24
3.3	The Remaining Gauge Fields	26
3.4	Ambiguous Terms	30
3.5	Conformal Properties of the Effective Action	31
4	Perturbative Treatment of the Matter Fields	33
4.1	One-Loop Effects and Bosonization	35
4.1.1	Zeta Function Regularization and the Heat Kernel . . .	37
4.1.2	Conformal Anomaly	39
4.1.3	Chiral Anomaly	40
4.2	Lowest Order Vertices	43
4.2.1	Effective Geometry and the Virtual Black Hole	43
4.2.2	Four-Point Vertices	48
4.3	Asymptotic Matter States	49

5	Conclusions and Possible Further Developments	53
5.1	Summary and Conclusions	53
5.1.1	Classical Analysis	53
5.1.2	Nonperturbative Quantization of Geometry	54
5.1.3	Matter Perturbation Theory and Bosonization	55
5.2	Possible Extensions	57
A	Notational Conventions	59
B	Details from Ch. 2	61
B.1	Poisson Brackets of the Secondary with the Second Class Con- straints	61
B.2	Dirac Brackets used in Sec. 2.2.3	64
B.3	Calculations from Sec. 2.3	65
	Bibliography	67

Chapter 1

Introduction

Gleichwohl müßten die Atome zufolge der inneratomischen Elektronenbewegung nicht nur elektromagnetische, sondern auch Gravitationsenergie ausstrahlen, wenn auch in winzigem Betrage. Da dies in Wahrheit in der Natur nicht zutreffen dürfte, so scheint es, daß die Quantentheorie nicht nur die Maxwellsche Elektrodynamik, sondern auch die neue Gravitationstheorie wird modifizieren müssen.

Albert Einstein [1]

Today, ninety years after Albert Einstein’s discovery of General Relativity [2, 3] and eighty years after the advent of quantum mechanics with the seminal papers by Heisenberg and Schrödinger [4, 5], an end of the search for a consistent and unique quantum theory of gravity is still not in sight. Many different approaches (cf. e.g. the recent progress report [6]) are available, amongst which the most prominent ones are loop quantum gravity and string theory, all stressing different aspects and succeeding in solving different problems while at the same time suffering from different drawbacks. The failure of our efforts in quantizing gravity until now can be traced back to conceptual and partly to technical issues. Concerning the conceptual problems, the foundations of general relativity, namely a dynamical space-time with no preferred reference frame, seem to be incompatible with the needs of quantum theory, which in its Hamiltonian formulation requires a choice of a preferred time direction and thus violates general covariance. Also the notion of causality in quantum field theory becomes meaningless if the metric of the underlying space-time is subject to quantum fluctuations which so to speak “smear out” the causal structure. On a deeper level, quantum gravity requires the quantization of space-time itself, a notion which first has to be given proper meaning by stating which the valid observables are for such a quantum space-time. Technical difficulties arise mostly from the already

complicated and highly nonlinear structure of general relativity. But if the classical dynamics of gravity already is this complicated, one can not expect quantization to be a much easier task.

One way to a better understanding of the conceptual issues behind quantum gravity is to consider gravitation in lower dimensions to simplify the dynamics of the theory. Experience with quantum field theory on Minkowski space shows that models in two dimensions often exhibit integrable or even topological structures, which makes them particularly interesting to study. Unfortunately, Einstein-Hilbert gravity in two dimensions is too simple - the action functional is just the Gauss-Bonnet term yielding the Euler number of the space-time manifold. The simplest way to generalize the Hilbert action by introducing an additional scalar field – the dilaton – leads to Generalized Dilaton Theories described by the action (2.1), which was first introduced in full generality in [7–9].¹ In their first order formulation described by the action (2.3), which was first given in this general form in [11,12], the dynamics of this class of theories indeed simplifies such that all classical solutions can be obtained explicitly [13] and a global classification of all generalized dilaton theories can be given [14]. In this context a background independent [15] and nonperturbative path integral quantization of first order gravity without matter [16], coupled to real scalar fields [17–19], of dilaton supergravity in two dimensions [20,21] and of dilaton gravity in Euclidean space-time [22] is possible. Also the role of time was discussed [23] in the context of Dirac quantization of dilaton gravity [24–27]².

In this thesis I consider the exact path integral quantization of first order gravity in two dimensions coupled to Dirac fermions.³ Interesting by itself because it has been a blind spot in the literature on quantum dilaton gravity until now, the main motivation of this analysis is the comparison to the scalar case [17–19]. The quantum equivalence between the massive Thirring model and the Sine-Gordon model in flat Minkowski space [34,35] has recently been applied [36–38] to semi-classical studies of two-dimensional dilaton gravity electrodynamics, with the matter model treated as a quantized theory on a classical background described by the dilaton gravity theory. As it is already known how to nonperturbatively quantize dilaton gravity coupled to scalar fields in the path integral formalism, the study of the corresponding fermionic systems is a prerequisite for investigating whether and how bosonization could carry over to the quantum (dilaton) gravity regime.

¹For a concise history of the topic cf. e.g. the review [10].

²These are only the most influential papers on that topic. A complete list of references can be found in chapter 9 of [10].

³Such models were first considered on the level of classical solutions [28–30] and used later on in studies of the evaporation of charged CGHS [31] black holes [32,33].

The structure of this thesis is as follows:

- Chapter 2 presents the constraint analysis of two-dimensional dilaton gravity in its first order formulation coupled to Dirac fermions. After calculating the algebra of first class constraints the BRST charge is constructed, the phase space is enlarged by a ghost sector and Eddington-Finkelstein gauge is fixed by constructing a BRST invariant Hamiltonian.
- In chapter 3 the first order gravity sector of the theory is quantized nonperturbatively using the Feynman path integral approach, resulting in an effective action which still depends on the Dirac fermions.
- The remaining path integral quantization of the Dirac fermions is carried out by means of perturbation theory in chapter 4. For massless and minimally coupled fermions chiral and conformal anomalies are used to calculate the one-loop approximation of the total effective action of the system. The gravitationally induced four-fermi scattering vertices are derived.
- Chapter 5 contains a summary of the results, conclusions and an outlook to possible further developments.

Chapter 2

Classical Analysis

2.1 Dilaton Gravity in Two Dimensions with Fermions

The field theoretic system considered in this thesis consists of two sectors, a geometric one and a matter sector. The geometric sector consists of a scalar field X called the dilaton field and the metric $g_{\mu\nu}$ on a (1+1) dimensional space-time manifold with signature $(+ -)$ and is described by the Generalized Dilaton Gravity action

$$S_{\text{2DG}} = \frac{1}{2} \int d^2x \sqrt{-g} \left[X R - U(X) (\nabla X)^2 + 2V(X) \right], \quad (2.1)$$

with R being the Ricci scalar associated with the metric $g_{\mu\nu}$ and ∇ the Levi-Civita connection. The functions $U(X)$ and $V(X)$ are specifying the model under consideration. Table 2.1 gives a list of models fitting into this action. Introducing a Zweibein e_μ^a and a dual dyad 1-form $e^a = e_\mu^a dx^\mu$ which are related to each other and to the metric by¹

$$g_{\mu\nu} = e_\mu^a \eta_{ab} e_\nu^b, \quad e_\mu^a e_b^\mu = \delta_b^a, \quad e_a^\mu e_\nu^\mu = \delta_\nu^a, \quad (2.2)$$

the first order gravity action

$$S^{\text{FOG}} = \int_{\mathcal{M}_2} [X_a T^a + X \mathcal{R} + \epsilon \mathcal{V}(X^a X_a, X)] \quad (2.3)$$

is equivalent to (2.1) for potentials [10]

$$\mathcal{V}(X^a X_a, X) = \frac{X^a X_a}{2} U(X) + V(X). \quad (2.4)$$

¹For conventions used throughout this thesis, see App. A.

Model (cf. (2.1) or (2.3))	$U(X)$	$V(X)$	$w(X)$ (cf. (2.34))
1. Schwarzschild [39]	$-\frac{1}{2X}$	$-\lambda^2$	$-2\lambda^2\sqrt{X}$
2. Jackiw-Teitelboim [40, 41]	0	$-\Lambda X$	$-\frac{1}{2}\Lambda X^2$
3. Witten BH/CGHS [31, 42]	$-\frac{1}{X}$	$-2b^2 X$	$-2b^2 X$
4. CT Witten BH [31, 42]	0	$-2b^2$	$-2b^2 X$
5. SRG ($D > 3$)	$-\frac{D-3}{(D-2)X}$	$-\lambda^2 X^{(D-4)/(D-2)}$	$-\lambda^2 \frac{D-2}{D-3} X^{(D-3)/(D-2)}$
6. $(A)dS_2$ ground state [43]	$-\frac{a}{X}$	$-\frac{B}{2} X$	$a \neq 2 : -\frac{B}{2(2-a)} X^{2-a}$
7. Rindler ground state [44]	$-\frac{a}{X}$	$-\frac{B}{2} X^a$	$-\frac{B}{2} X$
8. BH attractor [45]	0	$-\frac{B}{2} X^{-1}$	$-\frac{B}{2} \ln X$
9. All above: ab -family [46]	$-\frac{a}{X}$	$-\frac{B}{2} X^{a+b}$	$b \neq -1 : -\frac{B}{2(b+1)} X^{b+1}$
10. Liouville gravity [47]	a	$be^{\alpha X}$	$a \neq -\alpha : \frac{b}{a+\alpha} e^{(a+\alpha)X}$
11. Scattering trivial [48]	generic	0	const.
12. Reissner-Nordström [49]	$-\frac{1}{2X}$	$-\lambda^2 + \frac{Q^2}{X}$	$-2\lambda^2\sqrt{X} - 2Q^2/\sqrt{X}$
13. Schwarzschild- $(A)dS$ [50]	$-\frac{1}{2X}$	$-\lambda^2 - \ell X$	$-2\lambda^2\sqrt{X} - \frac{2}{3}\ell X^{3/2}$
14. Katanaev-Volovich [51]	α	$\beta X^2 - \Lambda$	$\int^X e^{\alpha y} (\beta y^2 - \Lambda) dy$
15. Achúcarro-Ortiz [52]	0	$\frac{Q^2}{X} - \frac{J}{4X^3} - \Lambda X$	$Q^2 \ln X + \frac{J}{8X^2} - \frac{1}{2}\Lambda X^2$
16. KK reduced CS [53, 54]	0	$\frac{1}{2}X(c - X^2)$	$-\frac{1}{8}(c - X^2)^2$
17. Symmetric kink [55]	generic	$-X \prod_{i=1}^n (X^2 - X_i^2)$	cf. [55]
18. 2D type 0A/0B [56, 57]	$-\frac{1}{X}$	$-2b^2 X + \frac{b^2 q^2}{8\pi}$	$-2b^2 X + \frac{b^2 q^2}{8\pi} \ln X$
19. exact string BH [58, 59]	cf. [59]	cf. [59]	cf. [59]

Table 2.1: List of models (taken from [60])

The quantity ϵ denotes the volume form on the manifold. The fields X^a are Lagrange multipliers for the torsion 2-form, which in turn is defined by Cartan's first structure equation

$$T^a = D^a_b e^b = (\delta^a_b d + \epsilon^a_b \omega) \wedge e^b, \quad (2.5)$$

with $D^a_b = \delta^a_b d + \epsilon^a_b \omega$ being the local Lorentz covariant derivative written as a 1-form. The spin connection is required to be metric compatible and thus antisymmetric [61], $\omega^a_b = -\omega_b^a$, but first order gravity allows for torsion if $U(X) \neq 0$. Because the Lorentz group in two dimensions, $SO(1, 1)$, has only one generator, ϵ^a_b , the spin connection can then be expressed as $\omega^a_b = \epsilon^a_b \omega$. The quantity \mathcal{R} in (2.3) is the curvature 2-form related to the spin connection by Cartan's second structure equation

$$\mathcal{R}^a_b = D^a_c \omega^c_b = \epsilon^a_b d\omega + \epsilon^a_c \epsilon^c_b \omega \wedge \omega = \epsilon^a_b d\omega \quad (2.6)$$

$$\mathcal{R} := \frac{1}{2} \epsilon^a_b \mathcal{R}^b_a = d\omega. \quad (2.7)$$

It is related to the Ricci scalar by $R = 2 * d\omega$. Using (A.4) and (A.5) this relation is inverted, yielding

$$d\omega = \frac{R}{2}\epsilon. \quad (2.8)$$

This Ricci scalar however differs from the one in eq. (2.1) by the torsion part of the spin connection. Only for theories with vanishing torsion $U(X) = 0$ both definitions coincide (also cf. the remarks in sec. 4.2.1 on eqs. (4.88) and (4.93)).

2.1.1 Equivalence between First and Second Order Formulation

The equivalence between first order gravity (2.3) and generalized dilaton theories (2.1) is established in two steps [10]. First one decomposes the spin connection into a Levi-Civita part $\tilde{\omega}$ and a torsion part ω_T ,

$$\omega = \tilde{\omega} + \omega_T. \quad (2.9)$$

The Levi-Civita spin connection explicitly reads

$$\tilde{\omega} = e_a(*de^a), \quad (2.10)$$

has vanishing torsion $\tilde{T}^a = (\tilde{D}e)^a = 0$ and will be considered fixed from now on. The second step is to use the equations of motion following from varying eq. (2.3) with respect to X^a and ω_T ,

$$0 = \underbrace{(\tilde{D}e)^a}_{=0} + \epsilon^a_b \omega_T \wedge e^b + \epsilon \frac{\partial \mathcal{V}}{\partial X_a} \quad (2.11)$$

$$0 = dX + X^a \epsilon_{ab} e^b, \quad (2.12)$$

to eliminate both the Lagrange multipliers X^a and the torsion part ω_T of the spin connection. Wedge multiplying eq. (2.12) with e^c from the right, using eq. (A.9) and taking the Hodge dual yields the solution for the Lagrange multipliers

$$X^a = *(e^a \wedge dX). \quad (2.13)$$

Decomposing $\omega_T = \omega_{Ta}e^a$, a similar calculation for the class of potentials (2.4) yields from eq. (2.11)

$$\omega_T = U(X)X_a e^a = U(X) * (e_a \wedge dX) e^a. \quad (2.14)$$

The torsion part of the spin connection thus vanishes if and only if $U(X) = 0$. Inserting eq. (2.9) into (2.3) gives after using the torsionlessness of the Levi-Civita connection, eq. (2.8) and a partial integration

$$S_1^{\text{FOG}} = \int_{\mathcal{M}_2} \left[\epsilon \frac{\tilde{R}}{2} X + \epsilon V(X) - dX \wedge \omega_T + X^a \epsilon_{ab} \omega_T \wedge e^b + \epsilon \frac{U(X)}{2} X^a X_a \right]. \quad (2.15)$$

After inserting eq. (2.14) into eq. (2.15) and using

$$X^a X_a = -g^{\mu\nu} (\nabla_\mu X) (\nabla_\nu X),$$

the third and fourth term in (2.15) cancel and the last one yields the kinetic term for the dilaton in (2.1). This proves the equivalence of the first order gravity action eq. (2.3) and the dilaton gravity action eq. (2.1) for potentials of the form eq. (2.4). The above calculation also holds for first order gravity coupled to matter fields which do neither couple to the Lagrange multipliers X^a nor to the spin connection ω , so that the equations of motion (2.11) and (2.12) do not receive additional matter contributions. It holds in particular for fermions which in general do couple to the spin connection through their kinetic term, but not in dimension two (cf. sec. 2.1.4). Under the same restrictions the equivalence proof given in [16] for the corresponding quantum theories without matter also applies to the case with matter.

2.1.2 Classical Solutions

First order gravity (2.3) is an integrable system [10], i.e. all classical solutions can be obtained in a closed form. Although integrability is lost in general if matter is added, only one integration cannot be carried out explicitly. First denote the variations of the matter Lagrangian by

$$W^\pm := \frac{\delta S^{\text{m}}}{\delta e^\mp}, \quad W := \frac{\delta S^{\text{m}}}{\delta X}, \quad (2.16)$$

where the indices \pm denote light cone coordinates (A.1) in tangent and cotangent space. A coupling of matter to the spin connection is not considered, because neither fermions in two dimensions nor scalars do couple in such a way. The equations of motion corresponding to variations $\delta\omega$, δe^\mp , δX and δX^\mp respectively are

$$0 = dX + X^- e^+ - X^+ e^- \quad (2.17)$$

$$0 = (d \pm \omega) X^\pm \mp \mathcal{V} e^\pm + W^\pm \quad (2.18)$$

$$0 = d\omega + \epsilon \frac{\partial \mathcal{V}}{\partial X} + W \quad (2.19)$$

$$0 = (d \pm \omega) e^\pm + \epsilon \frac{\partial \mathcal{V}}{\partial X^\mp}. \quad (2.20)$$

In a patch with $X^+ \neq 0$ a new one-form $Z = e^+/X^+$ can be introduced and eq. (2.18) with the upper sign gives

$$\omega = -\frac{dX^+}{X^+} + Z\mathcal{V} - \frac{W^+}{X^+}, \quad (2.21)$$

while eq. (2.17) yields

$$e^- = \frac{dX}{X^+} + X^- Z. \quad (2.22)$$

Using these two solutions plus eq. (2.20) with the upper sign (and $\epsilon = e^+ \wedge e^- = Z \wedge dX$) one derives the relation

$$dZ - U(X)dX \wedge Z = \frac{W^+}{X^+} \wedge Z. \quad (2.23)$$

If $W^+ = 0$, which corresponds to (anti)chiral fermions (cf. eq. (2.50)) and (anti)self-dual scalars², the Ansatz $Z = \hat{Z}e^{Q(X)}$ (with $Q(X)$ defined in (2.35) below) leads to $d\hat{Z} = 0$, which can be integrated by use of the Poincaré Lemma to $\hat{Z} = df$. This exact integration is not possible for general matter contributions. The classical solutions obtained in this way are

$$e^+ = X^+ e^{Q(X)} df \quad (2.24)$$

$$e^- = \frac{dX}{X^+} + X^- e^{Q(X)} df \quad (2.25)$$

$$\omega = -\frac{dX^+}{X^+} + e^{Q(X)} \mathcal{V} df. \quad (2.26)$$

The theory exhibits an absolute, i.e. in space and time, conserved quantity [13, 62, 63] (with $w(X)$ defined in (2.34) below)

$$d\mathcal{C} = 0 \quad (2.27)$$

$$\mathcal{C} = \mathcal{C}^g + \mathcal{C}^m \quad (2.28)$$

$$= e^{Q(X)} X^+ X^- + w(X) + \mathcal{C}^m = \mathcal{C}_0 = \text{const.} \quad (2.29)$$

$$d\mathcal{C}^m = e^{Q(X)} (X^+ W^- + X^- W^+). \quad (2.30)$$

The line element for the solutions reads

$$(ds)^2 = \eta_{ab} e^a \otimes e^b = 2e^{Q(X)} df \otimes [dX + X^+ X^- e^{Q(X)} df]. \quad (2.31)$$

²The (anti)self-dual components of a real scalar field ϕ are defined by $\phi^\pm = *(\mathrm{d}\phi \wedge e^\pm)$. In these terms the action for a non-minimally coupled free scalar reads $S_\phi = \int d^2x \sqrt{-g} f(X) \phi^+ \phi^-$.

The quantity $X^+X^-e^{Q(X)}$ encodes information about the horizons of the solutions, which lie at $X^+X^- = 0$. In the absence of matter $W^\pm = W = 0$ it can with the help of the conserved quantity be rewritten as $\mathcal{C}_0 - w(X)$, which immediately shows the existence of a Killing vector $\partial/\partial f$ for all dilaton gravity models. The corresponding Killing norm reads $e^{Q(X)}(\mathcal{C}_0 - w(X))$.

For the sake of completeness it should be mentioned that in addition to the vacuum solutions obtained above which are labeled by the conserved quantity \mathcal{C}^g , isolated solutions (cf. e.g. [55]) with constant dilaton $X = X_{\text{CDV}}$, so-called constant dilaton vacua (CDV), exist if X^\pm both vanish in some open region. Eq. (2.18) restricts the value of the dilaton to be one of the zeroes of the potential $V(X)$, $V(X_{\text{CDV}}) = 0$. The curvature for these solutions is constant (cf. eq. (2.19)),

$$d\omega = -\epsilon V'(X_{\text{CDV}}), \quad R = 2 * d\omega = -2V'(X_{\text{CDV}}). \quad (2.32)$$

The geometry of the constant dilaton vacua is thus Anti-de Sitter space ($R > 0$), de Sitter space ($R < 0$) and Minkowski or Rindler space ($R = 0$).³

2.1.3 Conformally Related Theories

Although first order gravity (2.3) is not conformally invariant, dilaton dependent conformal transformations

$$X^a \mapsto \frac{X^a}{\Omega}, \quad e^a \mapsto e^a \Omega, \quad \omega \mapsto \omega + X_a e^a \frac{d \ln \Omega}{dX} \quad (2.33)$$

with a conformal factor $\Omega = e^{\frac{1}{2} \int^X (U(y) - \tilde{U}(y)) dy}$ map a model with potentials $(U(X), V(X))$ to one with⁴ $(\tilde{U}(X), \tilde{V}(X) = \Omega^2 V(X))$. Thus one can always transform to a conformal frame with $\tilde{U} = 0$, which simplifies the equations of motion considerably, do calculations there and afterwards transform back to the original conformal frame. The expression

$$w(X) = \int^X e^{Q(y)} V(y) dy \quad (2.34)$$

is invariant under conformal transformations, whereas

$$Q(X) = \int^X U(y) dy \quad (2.35)$$

captures the information about the conformal frame.

³In the conventions of [10], which are adopted in this thesis, the sphere has negative curvature.

⁴The action picks up a boundary term $\frac{1}{2} \int_{\mathcal{M}_2} d(X X_a e^a (U(X) - \tilde{U}(X)))$.

2.1.4 Fermions

The action for a Dirac fermion in two dimensions (see App. A for conventions) consists of a kinetic term⁵

$$S^{(\text{kin})} = \frac{i}{2} \int_{\mathcal{M}_2} f(X) (*e^a) \wedge (\overline{\chi} \gamma_a \overleftrightarrow{d} \chi), \quad (2.36)$$

and a general self-interaction

$$S^{(\text{SI})} = - \int_{\mathcal{M}_2} \epsilon h(X) g(\overline{\chi} \chi). \quad (2.37)$$

The functions $f(X)$ and $h(X)$ are the dilaton coupling functions. If they are constant one speaks of minimally coupled matter, and of non-minimally coupled matter otherwise. Because of the Grassmann property of the spinor field the self-interaction can in two dimensions at most include a mass term and a Thirring term,

$$g(\overline{\chi} \chi) = m \overline{\chi} \chi + \lambda (\overline{\chi} \chi)^2. \quad (2.38)$$

The constant contribution in this Taylor expansion of g has been set to zero because it would only shift the dilaton potential $V(X)$.

In two dimensions the kinetic term is independent of the spin connection. In arbitrary dimension, the kinetic term for fermions on a curved background reads⁶ [65]

$$\frac{i}{2} \int d^n x \det(e_\mu^a) [e_a^\mu (\overline{\chi} \gamma^a (\partial_\mu + \omega_\mu^{bc} \Sigma_{bc}) \chi) + \text{h.c.}] .$$

For $n = 2$ there is only one independent generator of Lorentz transformations $\Sigma_{01} = \frac{1}{4}[\gamma_0, \gamma_1] = \frac{\gamma_*}{2}$, and with $\{\gamma_a, \gamma_*\} = 0$ the terms in (2.36) containing the spin connection vanish,

$$\begin{aligned} & \frac{i}{4} (*e^a) \wedge \omega \chi^\dagger (\gamma^0 \gamma_a \gamma_* - \gamma_* \gamma_a^\dagger \gamma^0) \chi \\ &= \frac{i}{4} (*e^a) \wedge \omega \chi^\dagger \underbrace{(\gamma^0 \gamma_a - \gamma_a^\dagger \gamma^0)}_{=0} \gamma_* \chi = 0. \end{aligned}$$

This simplifies the constraint algebra and is one of the reasons why the path integral over the geometric sector can be carried out explicitly. In higher dimensions there would be a multiplicative coupling between the Vielbeine and the spin connection. Only in two dimensions the whole Lagrangian turns out to be at most linear in the geometric fields e^a and ω .

⁵The derivative acting on both sides is defined as $A \overleftrightarrow{\partial} B = A \partial B - (\partial A) B$. The sign difference between [64] and this thesis for $S^{(\text{kin})}$ stems from the differing sign of ϵ^{ab} .

⁶The abbreviation “h.c.” means hermitian conjugate.

2.2 Hamiltonian Analysis

Henceforth we denote the canonical coordinates and momenta by

$$\bar{q}^i = (\omega_0, e_0^-, e_0^+) \quad (2.39)$$

$$q^i = (\omega_1, e_1^-, e_1^+), \quad i = 1, 2, 3 \quad (2.40)$$

$$p_i = (X, X^+, X^-) \quad (2.41)$$

$$Q^\alpha = (\chi_0, \chi_1, \chi_0^*, \chi_1^*), \quad \alpha = 0, 1, 2, 3. \quad (2.42)$$

The canonical structure on the phase space is given by graded equal-time Poisson brackets

$$\begin{aligned} \{q^i(x), p_j(y)\} &= \delta_j^i \delta(x-y) \\ \{Q^\alpha(x), P_\beta(y)\} &= -\delta_\beta^\alpha \delta(x-y), \end{aligned} \quad (2.43)$$

where the P_β are canonical momenta for the spinors defined by $P_\beta := \frac{\partial^L \mathcal{L}}{\partial Q^\beta}$ (∂^L is the usual left derivative on Grassmann numbers). The graded Poisson bracket is defined as [66]

$$\begin{aligned} \{F, G\} &= \int dz \left[\left(\frac{\partial F}{\partial q^i(z)} \frac{\partial G}{\partial p_i(z)} - \frac{\partial F}{\partial p_i(z)} \frac{\partial G}{\partial q^i(z)} \right) \right. \\ &\quad \left. + (-)^{\varepsilon(F)} \left(\frac{\partial^L F}{\partial Q^\alpha(z)} \frac{\partial^L G}{\partial P_\alpha(z)} - \frac{\partial^L F}{\partial P_\alpha(z)} \frac{\partial^L G}{\partial Q^\alpha(z)} \right) \right], \end{aligned} \quad (2.44)$$

with (q^i, p_i) and (Q^α, P_α) being bosonic ($\varepsilon(q^i) = \varepsilon(p_i) = 0$) and fermionic ($\varepsilon(Q^\alpha) = \varepsilon(P_\alpha) = 1$) canonical pairs, respectively. F and G are functions on the phase space with definite Grassmann parities $\varepsilon(F), \varepsilon(G)$. In this chapter three main properties of this bracket will be frequently used, namely the symmetry property, the Leibniz rule and the graded Jacobi identity

$$\{F, G\} = (-)^{\varepsilon(F)\varepsilon(G)+1} \{G, F\} \quad (2.45)$$

$$\{F, G_1 G_2\} = \{F, G_1\} G_2 + (-)^{\varepsilon(F)\varepsilon(G_1)} G_1 \{F, G_2\} \quad (2.46)$$

$$\begin{aligned} 0 &= (-)^{\varepsilon(F_1)\varepsilon(F_3)} \{\{F_1, F_2\}, F_3\} \\ &\quad + (-)^{\varepsilon(F_1)\varepsilon(F_2)} \{\{F_2, F_3\}, F_1\} \\ &\quad + (-)^{\varepsilon(F_2)\varepsilon(F_3)} \{\{F_3, F_1\}, F_2\}. \end{aligned} \quad (2.47)$$

Henceforth the graded Poisson bracket will just be referred to as ‘‘Poisson bracket’’ except for where the distinction is crucial.

2.2.1 Primary and Secondary Constraints

The system under consideration admits both primary first and second class constraints. A look at the component form of the Lagrangian,

$$\mathcal{L} = \mathcal{L}^{\text{FOG}} + \mathcal{L}^{(\text{kin})} + \mathcal{L}^{(\text{SI})} \quad (2.48)$$

$$\begin{aligned} \mathcal{L}^{\text{FOG}} = & \tilde{\epsilon}^{\nu\mu} (X^+ (\partial_\mu - \omega_\mu) e_\nu^- + X^- (\partial_\mu + \omega_\mu) e_\nu^+ \\ & + X \partial_\mu \omega_\nu) + (e) \mathcal{V}(X^+ X^-; X) \end{aligned} \quad (2.49)$$

$$\begin{aligned} \mathcal{L}^{(\text{kin})} = & \frac{i}{\sqrt{2}} f(X) \left[-e_0^+ (\chi_0^* \overleftrightarrow{\partial}_1 \chi_0) + e_0^- (\chi_1^* \overleftrightarrow{\partial}_1 \chi_1) \right. \\ & \left. + e_1^+ (\chi_0^* \overleftrightarrow{\partial}_0 \chi_0) - e_1^- (\chi_1^* \overleftrightarrow{\partial}_0 \chi_1) \right] \end{aligned} \quad (2.50)$$

$$\mathcal{L}^{(\text{SI})} = -(e) h(X) g(\chi_1^* \chi_0 + \chi_0^* \chi_1), \quad (2.51)$$

where $(e) = \det e_\mu^a$, shows that there do not occur any time (i.e. x^0) derivatives of the \bar{q}^i and thus the corresponding canonical momenta $\bar{p}_i := \frac{\partial \mathcal{L}}{\partial \bar{q}^i}$ are constrained to zero,

$$\bar{p}_i \approx 0. \quad (2.52)$$

The symbol \approx denotes weak equality, i.e. equality on the constraint surface. Because the kinetic term for the fermions is of first order in time derivatives, the fermion momenta P_α are related to the spinor components by another set of primary constraints,

$$\Phi_0 = P_0 + \frac{i}{\sqrt{2}} f(p_1) q^3 Q^2 \approx 0 \quad (2.53)$$

$$\Phi_1 = P_1 - \frac{i}{\sqrt{2}} f(p_1) q^2 Q^3 \approx 0 \quad (2.54)$$

$$\Phi_2 = P_2 + \frac{i}{\sqrt{2}} f(p_1) q^3 Q^0 \approx 0 \quad (2.55)$$

$$\Phi_3 = P_3 - \frac{i}{\sqrt{2}} f(p_1) q^2 Q^1 \approx 0, \quad (2.56)$$

which have nonvanishing Poisson brackets with each other,

$$\begin{aligned} C_{\alpha\beta}(x, y) &:= \{\Phi_\alpha(x), \Phi_\beta(y)\} \\ &= i\sqrt{2} f(X) \begin{pmatrix} 0 & 0 & -e_1^+ & 0 \\ 0 & 0 & 0 & e_1^- \\ -e_1^+ & 0 & 0 & 0 \\ 0 & e_1^- & 0 & 0 \end{pmatrix} \delta(x - y), \end{aligned} \quad (2.57)$$

and thus are of second class according to Dirac's classification of constraints [67] if the matrix (2.57) has maximal rank. In Eddington-Finkelstein gauge

$(e_0^-, e_0^+) = (1, 0)$, which will become very important later on, the condition for a horizon is $e_1^- = 0$. The Dirac matrix (2.57) thus has maximal rank away from horizons and the four constraints (2.53)-(2.56) are all of second class. On horizons it has rank two, but the rank never vanishes for a nondegenerate metric. Thus on a horizon second class constraints are converted into first class ones, i.e. on the horizon more gauge symmetries are present than away from it. This phenomenon was first mentioned by t'Hooft [68] and appears in several settings, see [69] and references therein. As mentioned in sec. 5.2, it also seems to be connected to the problem of black hole universality.

The Φ_α are independent of the \bar{q}^i and thus commute even in the strong sense with the other three primary constraints \bar{p}_i ,

$$\{\bar{p}_i, \Phi_\alpha\} = 0. \quad (2.58)$$

Having computed all the momenta, one obtains the Hamiltonian density from the Lagrangian (2.48) by means of ordinary Legendre transformation,

$$\begin{aligned} \mathcal{H} &= \dot{Q}^\alpha P_\alpha + p_i \dot{q}^i - \mathcal{L} \\ &=: \mathcal{H}_{FOG} + \mathcal{H}_{kin} + \mathcal{H}_{SI} \end{aligned} \quad (2.59)$$

$$\begin{aligned} \mathcal{H}_{FOG} &= X^+(\partial_1 - \omega_1)e_0^- + X^-(\partial_1 + \omega_1)e_0^+ + X\partial_1\omega_0 - (e)\mathcal{V} \\ &\quad + (X^+e_1^- - X^-e_1^+)\omega_0 \end{aligned} \quad (2.60)$$

$$\mathcal{H}_{kin} = \frac{i}{\sqrt{2}}f(X) \left[e_0^+(\chi_0^* \overleftrightarrow{\partial}_1 \chi_0) - e_0^-(\chi_1^* \overleftrightarrow{\partial}_1 \chi_1) \right] \quad (2.61)$$

$$\mathcal{H}_{SI} = (e)h(X)g(\overline{\chi}\chi) \quad (2.62)$$

To deal with the second class constraints one introduces the Dirac bracket [66, 67]

$$\{F_x, G_y\}^* := \{F_x, G_y\} - \int dz dw \{F_x, \Phi_\alpha(z)\} C^{\alpha\beta}(z, w) \{\Phi_\beta(w), G_y\}, \quad (2.63)$$

with the subscripts x, y denoting functions evaluated at the corresponding points in space. It inherits the properties (2.45), (2.46) and (2.47) from the graded Poisson bracket. $C^{\alpha\beta}(x, y)$ is the inverse of the matrix-valued distribution (2.57) defined such that

$$\int dy \left(\int dx \varphi(x) C_{\alpha\gamma}(x, y) \right) \left(\int dz \psi(z) C^{\gamma\beta}(y, z) \right) = \delta_\alpha^\beta \int dx \varphi(x) \psi(x)$$

for all test functions φ, ψ . It reads explicitly

$$C^{\alpha\beta}(x, y) = \frac{i}{\sqrt{2}f(X)} \begin{pmatrix} 0 & 0 & \frac{1}{e_1^+} & 0 \\ 0 & 0 & 0 & -\frac{1}{e_1^-} \\ \frac{1}{e_1^+} & 0 & 0 & 0 \\ 0 & -\frac{1}{e_1^-} & 0 & 0 \end{pmatrix} \delta(x - y). \quad (2.64)$$

Demanding the primary first class constraints \bar{p}_i not to change during time evolution, i.e.⁷

$$G_i := \dot{\bar{p}}_i = \{\bar{p}_i, \mathcal{H}'\}^* = \{\bar{p}_i, \mathcal{H}'\} \approx 0,$$

yields secondary constraints

$$G_1 = G_1^g \tag{2.65}$$

$$G_2 = G_2^g + \frac{i}{\sqrt{2}} f(X) (\chi_1^* \overleftrightarrow{\partial}_1 \chi_1) + e_1^+ h(X) g(\bar{\chi} \chi) \tag{2.66}$$

$$G_3 = G_3^g - \frac{i}{\sqrt{2}} f(X) (\chi_0^* \overleftrightarrow{\partial}_1 \chi_0) - e_1^- h(X) g(\bar{\chi} \chi) \tag{2.67}$$

with G_i^g being the first order gravity constraints [16]

$$G_1^g = \partial_1 X + X^- e_1^+ - X^+ e_1^- \tag{2.68}$$

$$G_2^g = \partial_1 X^+ + \omega_1 X^+ - e_1^+ \mathcal{V} \tag{2.69}$$

$$G_3^g = \partial_1 X^- - \omega_1 X^- + e_1^- \mathcal{V}. \tag{2.70}$$

The Hamiltonian density turns out to be constrained to zero, as expected for a generally covariant system [66],

$$\mathcal{H} = -\bar{q}^i G_i, \tag{2.71}$$

where a boundary term $\int \partial_1(p_i \bar{q}^i)$ on the right hand side has been dropped. Working with smeared constraints ($\eta(x^1)$ is some test function)

$$\bar{p}_i[\eta] = \int dx^1 \eta \bar{p}_i \tag{2.72}$$

and with the Hamiltonian rather than the Hamiltonian density one finds that the secondary constraints (2.65)-(2.67) acquire boundary terms

$$G_i[\eta] = \int dx^1 G_i(x^1) \eta(x^1) - p_i \eta \Big|_{\partial \mathcal{M}_1}, \tag{2.73}$$

where $\partial \mathcal{M}_1$ denotes the boundaries of the x^1 -direction of space-time. The Hamiltonian with the \bar{q}^i playing the role of smearing functions turns out to be a sum over constraints even in the presence of boundaries,

$$H = \int dx^1 \mathcal{H} = -G_i[\bar{q}^i]. \tag{2.74}$$

⁷Henceforth, a prime in a Dirac or Poisson bracket means evaluation of the function at a point x' , whereas a prime on a function not being an argument of some bracket denotes differentiation with respect to the argument of the function.

This is true for first order gravity supplemented by a Gibbons-Hawking boundary term, as shown in [69], as well as without any additional boundary terms in the action.

The secondary constraints have vanishing Dirac bracket with the \bar{p}_i because both the G_i and Φ_α are independent of the \bar{q}^i . They also trivially commute with the primary second class constraints, $\{\Phi_\alpha, G'_j\}^* = 0$, because of the definition of the Dirac bracket. For the same reason the Φ_α do not give rise to new secondary constraints.

2.2.2 Algebra of Secondary Constraints

Dirac conjectured [67] that every first class constraint generates a gauge symmetry. The proof of this conjecture is possible in a very general setting [70], but making some additional assumptions (see paragraph 3.3.2 of [66]) to rule out “pathological” examples simplifies it. These assumptions are fulfilled in our case, because 1. every constraint belongs to a well defined generation; 2. the Dirac bracket ensures that the primary second class constraints do not generate new ones and, as will be seen below, the secondary constraints are of first class and thus there are no further constraints generated by them; and 3. every primary first class constraint \bar{p}_i generates exactly one G_i .

To show that the system does not admit any ternary constraints it is sufficient to show that the algebra of secondary constraints closes weakly, i.e.

$$\{G_i, G'_j\}^* = C_{ij}{}^k(x) G_k \delta(x - x') \approx 0, \quad (2.75)$$

and thus the secondary constraints are preserved under the time evolution generated through the Dirac bracket,

$$\dot{G}_i = \{G_i, \mathcal{H}'\}^* = -\bar{q}^{j'} \{G_i, G'_j\}^* \approx 0.$$

To calculate all the Dirac brackets, one first needs the Poisson brackets $\{\Phi_\alpha, G_j\}$. They are rather lengthy and thus listed in Appendix B.1, together with some remarks on the actual calculation. The resulting algebra of secondary constraints reads

$$\{G_i, G'_i\}^* = 0 \quad i = 1, 2, 3 \quad (2.76)$$

$$\{G_1, G'_2\}^* = -G_2 \delta \quad (2.77)$$

$$\{G_1, G'_3\}^* = G_3 \delta \quad (2.78)$$

$$\{G_2, G'_3\}^* = \left[-\sum_{i=1}^3 \frac{\partial \mathcal{V}}{\partial p_i} G_i + \left(gh' - \frac{h}{f} f' g' \cdot (\bar{\chi} \chi) \right) G_1 \right] \delta. \quad (2.79)$$

It should be noted that the algebra for minimally coupled fermions with zero mass and without self-interaction was already computed in [71]. The right hand sides vanish weakly, and thus the secondary constraints do not generate new constraints and the Dirac procedure stops at this level. The algebra still closes like in the case of a compact Lie groups, $[\delta_A, \delta_B] = f^C_{AB}(x)\delta_C$, but rather with structure functions than with constants. The second term in (2.79) deserves some remarks. Without it, the algebra would be just the one obtained for first order gravity without matter [16]. It vanishes for minimal coupling, i.e. for $h = f = \text{const}$. For $h \propto f$ and $g = m\bar{\chi}\chi + \lambda(\bar{\chi}\chi)^2$ it becomes proportional to $f'(g - g' \cdot \bar{\chi}\chi)G_1\delta$, thus a mass term does not change the constraint algebra at all. Furthermore, it does not contain derivatives of the matter field as opposed to the case of scalar matter (see eq. E.31 in [19]), where the additional contribution to $\{G_2, G_3\}$ is proportional to $\frac{f'}{f}\mathcal{L}_{\text{scalar}}$. It is noteworthy that if the x^1 -direction has a boundary, in the matterless theory only the bracket between the diffeomorphism constraints eq. (2.79) is modified [60], acquiring a boundary term of the form

$$X(U(X) + XU'(X))X^+X^- - (V(X) - XV'(X)) \quad (2.80)$$

which vanishes only for $V \propto X$ and $U \propto 1/X$, i.e. for models with an $(A)dS_2$ ground state (cf. the second, third and sixth entry in table 2.1).

In contrast to the matterless theory [72] the algebra generated by the G_i and p_i is no classical finite W-algebra⁸ anymore,

$$\{G_1, p'_1\}^* = 0 \quad (2.81)$$

$$\{G_{2/3}, p'_1\}^* = \pm p_{2/3}\delta = -\{G_1, p'_{2/3}\}^* \quad (2.82)$$

$$\{G_2, p'_2\}^* = \frac{i}{\sqrt{2}}\frac{f}{q^2}(Q^3\overleftrightarrow{\partial}_1 Q^1)\delta + \frac{1}{2}\frac{q^2}{q^3}h g' \cdot (\bar{\chi}\chi)\delta \quad (2.83)$$

$$\{G_3, p'_3\}^* = -\frac{i}{\sqrt{2}}\frac{f}{q^3}(Q^2\overleftrightarrow{\partial}_1 Q^0)\delta - \frac{1}{2}\frac{q^2}{q^3}h g' \cdot (\bar{\chi}\chi)\delta \quad (2.84)$$

$$\{G_3, p'_2\}^* = \left[\mathcal{V} - h g - \frac{1}{2}h g' \cdot (\bar{\chi}\chi)\right]\delta = -\{G_2, p'_3\}^*. \quad (2.85)$$

The constraints G_1 and $G_{2/3}$ on-shell generate local Lorentz transformations and diffeomorphisms, respectively. The right bracket in eq. (2.82) shows that the X^\pm transform as local Lorentz vectors under the action of G_1 ,

$$\delta_\gamma X^\pm = \gamma\{X^\pm, G'_1\}^* = \pm\gamma X^\pm\delta. \quad (2.86)$$

⁸A classical finite W-algebra is, according to [73], an algebra with generators W_α and Poisson brackets $\{W_\alpha, W_\beta\} = P_{\alpha\beta}(W_\gamma)$, where $P_{\alpha\beta}$ are certain polynomials in the generators. In the case considered here the Dirac bracket replaces the Poisson bracket.

Eqs. (2.77) and (2.78) show that G_2 and G_3 also form a local Lorentz vector with components $G^+ = G_2$, $G^- = G_3$. The action of

$$G_\mu = -e_{\mu a} G^a \quad (2.87)$$

on the dilaton is

$$\begin{aligned} \delta_\xi X(x') &= \xi^\mu \{X, G'_\mu\}^* = -\xi^\mu \left[e_\mu^+ \{X, G'_3\}^* + e_\mu^- \{X, G'_2\}^* \right] \\ &= \xi^\mu \left[e_\mu^- X^+ - e_\mu^+ X^- \right] \delta \\ &= \xi^\mu (\partial_\mu X) \delta. \end{aligned} \quad (2.88)$$

In the last step the equation of motion (2.17) has been used. This shows that G_μ on-shell is the generator of diffeomorphisms.

2.2.3 Relation to the Conformal Algebra

As first noted in [74], certain linear combinations of the G_i fulfil the classical Virasoro algebra. In that work first order gravity coupled to scalar matter was considered, but the same result holds for the case with fermionic matter. New generators

$$G = G_1, \quad H_{0/1} = q^1 G_1 \mp q^2 G_2 + q^3 G_3 \quad (2.89)$$

fulfil an algebra (with $\delta' = \frac{\partial \delta(x-x')}{\partial x'}$)

$$\begin{aligned} \{G, G'\}^* &= 0 & \{H_i, H'_i\}^* &= (H_1 + H'_1) \delta' \\ \{G, H'_i\}^* &= -G \delta' & \{H_0, H'_1\}^* &= (H_0 + H'_0) \delta'. \end{aligned} \quad (2.90)$$

The constraints H_i form the well-known conformal algebra. The total algebra is the semidirect product of the conformal algebra and an invariant abelian subalgebra generated by the local Lorentz generator G [75]. The diffeomorphism part of H_1 is just $-G_0$ from eq. (2.87) above. The Dirac brackets needed for calculating this algebra are listed in App. B.2. Defining light cone combinations

$$H^\pm = \frac{1}{2}(H_0 \pm H_1), \quad (2.91)$$

the conformal part of the algebra (2.90) reads

$$\{H^\pm, H^{\pm'}\}^* = \pm(H^\pm + H^{\pm'}) \delta' \quad \{H^+, H^{-'}\}^* = 0. \quad (2.92)$$

The Fourier modes L_k, \bar{L}_k defined by

$$H^+(x) = \int \frac{dk}{2\pi} L_k e^{ikx}, \quad H^-(x) = \int \frac{dk}{2\pi} \bar{L}_k e^{ikx} \quad (2.93)$$

then obey an algebra

$$\begin{aligned} \{L_k, L_m\}^* &= i(m-k)L_{k+m} \\ \{\bar{L}_k, \bar{L}_m\}^* &= -i(m-k)\bar{L}_{m+k} \\ \{L_k, \bar{L}_m\}^* &= 0 \end{aligned} \quad (2.94)$$

which after redefinition $\bar{L} \mapsto -\bar{L}$ becomes the classical Virasoro algebra. Canonical quantization, i.e. representing the Virasoro generators as operators and replacing the Dirac bracket by $-i[.,.]$ leads to the well-known Virasoro algebra with zero central charge [76]

$$[L_k, L_m] = (k-m)L_{k+m}. \quad (2.95)$$

2.3 BRST Gauge Fixing

The path integral for a system with gauge symmetries is in general ill-defined. In order to obtain a well-defined path integral one has to explicitly break gauge invariance by means of constructing an appropriate gauge fixed action and afterwards restore gauge independence of the correlation functions of physical fields by enlarging the phase space with unphysical ghost and antighost fields.

In order to obtain a gauge fixed action, we use the general method of Batalin, Vilkovisky and Fradkin [77–79] based on the BRST symmetry [66, 80]. We first construct the BRST charge Ω . The system has three gauge symmetries generated by the G_i , so the phase space has to be enlarged by three pairs of ghosts and antighosts

$$(c_i, p_i^c) \quad i = 1, 2, 3, \quad (2.96)$$

and equipped with a graded Poisson bracket obeying the (anti)commutation relations (2.43) and

$$\{c^i, p_j^{c'}\} = -\delta_j^i \delta(x - x') \quad (2.97)$$

for the (anti)ghosts. The Dirac bracket is still defined as in (2.63), but now with the Poisson bracket of the extended phase space. One important point is that the ghosts themselves do not give rise to new second class constraints, thus the matrices (2.57), (2.64) are the same as for the gauge invariant system. The BRST charge acts on functions on the enlarged phase space through the Dirac bracket,

$$(\Omega F)(q, p, Q, P, c, p^c) := \{\Omega, F\}^*, \quad (2.98)$$

and has to be nilpotent,

$$\Omega^2 F = 0. \quad (2.99)$$

By virtue of the graded Jacobi identity (2.47) this is equivalent to

$$\{\Omega, \Omega\}^* = 0. \quad (2.100)$$

Furthermore Ω is required to act on functions depending only on the original phase space variables $(q^i, p_i, Q^\alpha, P_\alpha)$ like the original gauge transformations, i.e. through the generators G_i , and to have ghost number one. This means that one assigns $\text{gh}(c^i) = +1$ and $\text{gh}(p_j^c) = -1$ for ghosts and antighosts, respectively, and zero to all other phase space variables. The ghost number is additive for products of field monomials, $\text{gh}(AB) = \text{gh}(A) + \text{gh}(B)$. This leads to the Ansatz

$$\Omega = \underbrace{c^i G_i}_{\Omega^{(0)}} + \text{higher ghost terms}. \quad (2.101)$$

The first term in the above Ansatz gives (cf. (2.75))

$$\{\Omega^{(0)}, \Omega^{(0)'}\}^* = c^i c^j C_{ij}{}^k G_k \delta(x - x'). \quad (2.102)$$

The next higher term with total ghost number one is

$$\Omega^{(1)} = \frac{1}{2} c^i c^j C_{ij}{}^k p_k^c,$$

which cancels the contribution from the zeroth order,

$$\{\Omega^{(0)}, \Omega^{(1)'}\}^* + \{\Omega^{(1)}, \Omega^{(0)'}\}^* = -\{\Omega^{(0)}, \Omega^{(0)'}\}^* + D^{(1)}. \quad (2.103)$$

Detailed calculations can be found in App. B.3. The homological perturbation series terminates at Yang-Mills level⁹

$$\{\Omega^{(1)}, \Omega^{(1)'}\}^* = -D^{(1)}, \quad (2.104)$$

not because the structure functions commute with themselves but rather because the term containing the bracket of $C_{ij}{}^k$ with itself is quartic in the ghosts c^i , while the system contains only three anticommuting ghosts (2.96). The full BRST charge thus reads

$$\Omega = c^i G_i + \frac{1}{2} c^i c^j C_{ij}{}^k p_k^c. \quad (2.105)$$

⁹In Yang-Mills theory the structure functions $f_{ab}{}^c$ are just constants and thus the construction must necessarily terminate at this order, because $\{f_{ab}{}^c, f_{de}{}^f\} = 0$.

Constructed in this way it is unique up to canonical transformations of the extended phase space [66].

One now uses the theorem that BRST invariant functionals with total ghost number zero are sums of a BRST closed and a BRST exact part [80]. The gauge fixed Hamiltonian density should thus be of form

$$\mathcal{H}_{gf} = \mathcal{H}_{BRST} + \{\Omega, \Psi\}^*.$$

Choosing the gauge fixing fermion [19]

$$\Psi = p_2^c \quad (2.106)$$

and $\mathcal{H}_{BRST} = 0$ leads to Eddington-Finkelstein gauge

$$(\omega_0, e_0^-, e_0^+) = (0, 1, 0), \quad (2.107)$$

and to a gauge fixed Hamiltonian density (compare with (2.71))

$$\mathcal{H}_{gf} = \{\Omega, \Psi\}^* = -G_2 - C_{2i}^k c^i p_k^c. \quad (2.108)$$

The gauge fixed Lagrangian is obtained through Legendre transform in the extended phase space,

$$\begin{aligned} \mathcal{L}_{gf} &= \dot{Q}^\alpha P_\alpha + \dot{q}^i p_i + p_i^c \dot{c}^i - \mathcal{H}_{gf} \\ &= \dot{Q}^\alpha P_\alpha + \dot{q}^i p_i + G_2 + p_k^c M^k_l c^l, \end{aligned} \quad (2.109)$$

and contains the Faddeev-Popov operator

$$M = \begin{pmatrix} \partial_0 & 0 & \frac{\partial \mathcal{V}}{\partial X} - \left(gh' - \frac{h}{f} f' g' \cdot (\bar{\chi} \chi) \right) \\ -1 & \partial_0 & \frac{\partial \mathcal{V}}{\partial X^+} \\ 0 & 0 & \partial_0 + \frac{\partial \mathcal{V}}{\partial X^-} \end{pmatrix}. \quad (2.110)$$

The gauge choice eq. (2.107) still admits for some residual local Lorentz transformations and diffeomorphisms with parameters of the form

$$\gamma = \gamma(x^1), \quad \xi^0 = x^0 \gamma(x^1) + f(x^1), \quad \xi^1 = \xi^1(x^1), \quad (2.111)$$

as can be seen from the infinitesimal transformations of the gauged fields

$$0 = \delta \omega_0 = -\partial_0 \gamma + \omega_\nu \partial_0 \xi^\nu \quad (2.112)$$

$$0 = \delta e_0^- = -\gamma + e_\nu^- \partial_0 \xi^\nu \quad (2.113)$$

$$0 = \delta e_0^+ = e_\nu^+ \partial_0 \xi^\nu. \quad (2.114)$$

Non-infinitesimally these transformations are

$$\gamma = \gamma(x^1), \quad x^{0'} = e^{-\gamma(x^1)}(x^0 - g(x^1)), \quad x^{1'} = x^1(x^1). \quad (2.115)$$

Chapter 3

Nonperturbative Quantization of Geometry

In this chapter the path integral over the ghost sector (c^i, p_j^c) and the geometric variables (q^i, p_j) is evaluated nonperturbatively. The result is an effective action still depending on the fermions, which are quantized with perturbative methods in the next chapter.

3.1 Ghosts and Second Class Constraints

First, for later evaluation of correlation functions containing the geometric variables (q^i, p_j) , one couples them and the fermion field to external sources

$$\mathcal{L}_{\text{src}} = J^i p_i + j_i q^i + \bar{\eta} \chi + \bar{\chi} \eta. \quad (3.1)$$

The generating functional for correlation functions is formally given by the path integral

$$Z[J, j, \eta] = \mathcal{N} \int \mathcal{D}\mu[Q, P, q, p, c, p^c] e^{i \int d^2 x (\mathcal{L}_{gf} + \mathcal{L}_{\text{src}})} \quad (3.2)$$

with the actions (2.109) and (3.1), and the measure

$$\mathcal{D}\mu[Q, P, q, p, c, p^c] = \mathcal{D}\mu'[Q] \prod_{i=1}^3 \mathcal{D}p_i \mathcal{D}q^i \prod_{\alpha=0}^3 \mathcal{D}P_\alpha \delta(\Phi_\alpha) \prod_{j=1}^3 \mathcal{D}c^j \mathcal{D}p_j^c. \quad (3.3)$$

\mathcal{N} is a normalization factor. The delta functional in the measure [66] restricts the integration to the surface defined by the second class constraints eqs. (2.53)-(2.56), and the fermion measure $\mathcal{D}\mu'[Q]$ will be specified later.

The ghost integration can be carried out immediately, yielding the functional determinant of the Faddeev-Popov operator (2.110)

$$\Delta_{\Phi\Pi} = \text{Det}(\partial_0^2(\partial_0 + U(X)X^+)). \quad (3.4)$$

Integration of the fermion momenta P_α is trivial because of the delta functionals in (3.3) and the P_α -linearity of the second class constraints (2.53)-(2.56), yielding an effective Lagrangian

$$\mathcal{L}_{\text{eff}}^1 = p_i \dot{q}^i + G_2 + \frac{i}{\sqrt{2}} f(p_1) \left[q^3 (\chi_0^* \overleftrightarrow{\partial}_0 \chi_0) - q^2 (\chi_1^* \overleftrightarrow{\partial}_0 \chi_1) \right] + \mathcal{L}_{\text{src}}. \quad (3.5)$$

3.2 On the Path Integral Measure

Before integrating over (q^i, p_i) , one has to specify the integration measure for the matter fields. The reason is that for retaining general covariance of the measure [81–83] and thus preventing the diffeomorphism invariance of the classical field theory from acquiring an anomaly in the quantum theory, the matter fields in the integration measure have to be multiplied with appropriate powers of $\sqrt{-g}$, which in Eddington-Finkelstein gauge (2.107) is just $q^3 = e_1^+$. For the gauge-fixed path integral this means that the measure has to be invariant under those BRST transformations that generate general coordinate transformations [81]. As was proposed in [17], we use this choice of the measure because preserving general covariance also in the quantum theory is preferable from the physical point of view and also for subsequent application of known results, e.g. the derivation of one-loop effects in Ch. 4. The measure can be formally derived by analogy from the case of a finite dimensional vector space over complex Grassmann numbers [82, 83]. For Dirac fermions in two dimensions it reads (cf. eq. (7.14) in [83] with diffeomorphism weight $w = 0$)

$$\mathcal{D}\mu'[Q] = \prod_x [-g]^{-1} \mathcal{D}\bar{\chi} \mathcal{D}\chi = \prod_x [e_1^+]^{-2} \mathcal{D}\bar{\chi} \mathcal{D}\chi. \quad (3.6)$$

This procedure of changing the measure by hand to fit (3.6) can be implemented already at the level of the phase space path integral (3.2) where, knowing that the measure will be fixed by hand anyway, clandestinely a factor $\sqrt{\text{sdet} C_{\alpha\beta}}$ has been dropped.¹ This factor arises (see §16.1.1 of [66], [84]) from rewriting the path integral over the surface of second class constraints as a path integral over the whole phase space, with the aforementioned delta

¹Strictly speaking this should be a functional superdeterminant, but the functional part $\text{Det} \delta(x - y) = 1$ has already been separated.

functionals (cf. eq. (3.3)) again restricting the allowed paths to lie in the surface defined by second class constraints. The superdeterminant of a supermatrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

i.e. a matrix with even entries in the Bose-Bose part A and Fermi-Fermi part D and odd ones in the Bose-Fermi (Fermi-Bose) parts B (D), is defined via its supertrace [66]

$$\text{sdet} M = \exp \text{str} \ln M, \quad \text{str} M = \text{tr} A - \text{tr} D. \quad (3.7)$$

Keeping in mind that $C_{\alpha\beta}$ is a pure Fermi-Fermi matrix and thus

$$\text{sdet} C_{\alpha\beta} = \exp \left[-\text{tr} \ln C_{\alpha\beta} \right] = (\det C_{\alpha\beta})^{-1},$$

the measure factor becomes

$$\sqrt{\text{sdet} C_{\alpha\beta}} = (4f^4(X)(e_1^+ e_1^-)^2)^{-\frac{1}{2}}. \quad (3.8)$$

For minimal coupling $f(X) = 1$ and with the gauge choice (2.107) one arrives at the measure (3.6) in configuration space by starting with a phase space measure

$$\prod_x [-g^{00}(x)] \sqrt{\text{sdet} C_{\alpha\beta}(x)} \prod_{\alpha=0}^3 \mathcal{D}P_\alpha \quad (3.9)$$

instead of the one used in eq. (3.3). This is also what one would naively expect by analogy from the case of a real boson field [82] on a curved background, where a phase space measure $\mathcal{D}\mu[\pi, \phi] = \prod_x [-g^{00}(x)]^{-\frac{1}{2}} d\pi(x) d\phi(x)$

leads to the correct configuration space measure $\prod_x \sqrt[4]{-g(x)} d\phi(x)$ after integration over the canonical momenta. The measure factor for fermions in the Lagrangian path integral can be expected to be the inverse of the factors in the bosonic case, i.e. $[-g(x)]^{-1/4}$ for each real degree of freedom, because the volume element for Grassmann variables transforms inversely to that one for normal variables under changes of the basis. Indeed, the Dirac spinor has two complex and thus four real degrees of freedom, giving just the right factor (3.6). But the kinetic term for fermions (2.36) is already of first order in the time derivatives and thus fermions and their conjugate momenta are not independent in the Hamiltonian formalism. If one treats them as independent variables they only count as one-half degree of freedom each, yielding the right factor in (3.9).

For non-minimal coupling the question of which measure is the “right” covariant one is subtle and still not completely settled (for a review cf. e.g. [85]).

The additional measure factor (3.6) introduces a nonlinearity that prevents the immediate q^i -integration, but it can, following [18], formally be replaced by integration over an auxiliary field F and functional differentiating with respect to the current generating the field factor,

$$Z[J, j, \eta, \bar{\eta}] = \mathcal{N} \int \mathcal{D}F \delta \left(F - \frac{1}{i} \frac{\delta}{\delta j_3} \right) \tilde{Z}[F, J, j, \eta, \bar{\eta}] \quad (3.10)$$

$$\tilde{Z}[F, J, j, \eta, \bar{\eta}] = \int \mathcal{D}\bar{\chi} \mathcal{D}\chi \prod_x F^{-2} \prod_{i=1}^3 \mathcal{D}p_i \prod_{i=1}^3 \mathcal{D}q^i \Delta_{\Phi\Pi} e^{i \int d^2x \mathcal{L}_{\text{eff}}^{(1)}}. \quad (3.11)$$

3.3 The Remaining Gauge Fields

After eliminating the nonlinearity in the path integral measure, the q^i -linearity of (3.5) (cf. (2.66) and (2.69)) allows functional integration² over the q^i , yielding three delta functionals containing partial differential equations for the p_i ,

$$\partial_0 p_1 = j_1 + p_2 \quad (3.12)$$

$$\partial_0 p_2 = j_2 - \frac{i}{\sqrt{2}} f(p_1) (\chi_1^* \overleftrightarrow{\partial}_0 \chi_1) \quad (3.13)$$

$$(\partial_0 + U(p_1)p_2)p_3 = j_3 + \frac{i}{\sqrt{2}} f(p_1) (\chi_0^* \overleftrightarrow{\partial}_0 \chi_0) + h(p_1)g(\bar{\chi}\chi) - V(p_1). \quad (3.14)$$

These equations are the equations of motion for the p^i that result from the gauge fixed Lagrangian (2.109) after supplementing it by sources (3.1) and using the explicit expressions for the fermion momenta P_α which follow from the second class constraints (2.53)-(2.56).

Performing the p_i -integration is now equivalent to solving these equations for given currents j_i and matter fields, and substituting the solutions $p_i = \hat{B}_i$ back into the effective action obtained after the last step,

$$\mathcal{L}_{\text{eff}}^2 = J^i \hat{B}_i + \bar{\eta}\chi + \bar{\chi}\eta + \frac{i}{\sqrt{2}} f(\hat{B}_1) (\chi_1^* \overleftrightarrow{\partial}_1 \chi_1). \quad (3.15)$$

During this integration, the Faddeev-Popov determinant (3.4) also formally cancels, because the differential operators on the left hand side of eqs. (3.12) - (3.14) produce a factor $\text{Det}(\partial_0^2(\partial_0 + U(X)X^+))^{-1}$.

²A partial integration in the contact term in (3.5) produces a boundary contribution $\int d^2x \partial_0(p_i q^i)$, stemming from the unusual order of integration $q \rightarrow p$. Another one comes from (2.69), $\int d^2x \partial_1 X^+$, and is cancelled by the boundary term in eq. (2.73).

Eqs. (3.12) and (3.13) decouple for minimal coupling $f(X) = -\kappa = \text{const.}$ such that the solutions are formally

$$p_1 = \hat{B}_1 = \nabla_0^{-1}(j_1 + B_2) + \tilde{p}_1 \quad (3.16)$$

$$p_2 = \hat{B}_2 = \nabla_0^{-1} \left(j_2 + \kappa \frac{i}{\sqrt{2}} (\chi_1^* \overleftrightarrow{\partial}_0 \chi_1) \right) + \tilde{p}_2 \quad (3.17)$$

$$p_3 = \hat{B}_3 = e^{-\hat{Q}} \left[\nabla_0^{-1} e^{\hat{Q}} \left(j_3 - V(B_1) + h(B_1)g(\overline{\chi}\chi) - \kappa \frac{i}{\sqrt{2}} (\chi_0^* \overleftrightarrow{\partial}_0 \chi_0) \right) + \tilde{p}_3 \right]. \quad (3.18)$$

Here

$$\hat{Q}(\hat{B}_1, \hat{B}_2) := \nabla_0^{-1}(U(\hat{B}_1)\hat{B}_2) \quad (3.19)$$

is the regularized analogue of (2.35). For vanishing source $j_1 = 0$ it can be expressed (using (3.12)) as

$$\hat{Q}(\hat{B}_1) = \int^{\hat{B}_1} dy U(y), \quad (3.20)$$

where the unspecified lower integration limit means that (3.19) and (3.20) can differ by a homogeneous solution of the regularized time derivative ∇_0 . The quantities \tilde{p}_i are also homogeneous solutions of ∇_0 . Here the prescription of App. B of [18] is used, which provides an infrared regularization and proper asymptotic behaviour of the Green functions $\nabla_0^{-1}(x, y)$. The details of the regularization are, however, not important for what follows. In most calculations which explicitly use the regularized time derivative like e.g. [86] the important requirement is that $\nabla_0^{-1}f(x^1)$ should yield $f(x^1)x^0 + g(x^1)$ after undoing the regularization, i.e. the integral operator ∂_0^{-1} should act as an anti-derivate.

For general non-minimal coupling $f(X)$ eqs. (3.16) & (3.17) do not decouple, but still can be solved order by order with the Ansatz

$$p_i = \hat{B}_i = \sum_{n=0}^{\infty} p_i^{(n)} \quad i = 1, 2, \quad (3.21)$$

assuming the matter contributions in eq. (3.13) to be small, i.e. $p_1^{(n)}$ being of order n in fermion bilinears. Because the coupling functions $f(X), h(X)$ are of order of magnitude of the gravitational constant this approximation is valid for systems with total energies several orders of magnitude smaller than the Planck scale. Quantum gravity effects in few-particle scattering at energies accessible in present day particle accelerators are thus safely described in

this approach, but the approximation breaks down either for scattering at the Planck scale or for macroscopic matter accumulations like macroscopic black holes. In these cases a nonperturbative solution of eqs. (3.12) and (3.13) will be needed, which is only available for minimal coupling (eqs. (3.16) and (3.17)).

The lowest order contributions $p_i^{(0)}$ are then given by eqs. (3.16) & (3.17) with $\kappa = 0$. For linear dilaton coupling $f(X) = X$, the higher order terms are obtained through the recursion relations

$$p_2^{(n)} = -\frac{i}{\sqrt{2}} \nabla_0^{-1} \left(p_1^{(n-1)} (\chi_1^* \overleftrightarrow{\partial}_0 \chi_1) \right) \quad n \geq 1 \quad (3.22)$$

$$p_1^{(n)} = \nabla_0^{-1} p_2^{(n)}. \quad (3.23)$$

For general non-minimal couplings one needs to know the functional form of $f(X)$ to use this approach. A Taylor expansion around the zeroth term of eq. (3.21) yields

$$f(p_1) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p_1^{(0)})}{n!} \left(\sum_{m=1}^{\infty} p_1^{(m)} \right)^n. \quad (3.24)$$

One now has to truncate the expansion eq. (3.21) at some order $k \geq 1$ to be able to apply the binomial theorem. The zeroth order contributions are as above and the first order is

$$p_2^{(1)} = -\frac{i}{\sqrt{2}} \nabla_0^{-1} \left(f(p_1^{(0)}) (\chi_1^* \overleftrightarrow{\partial}_0 \chi_1) \right), \quad p_1^{(1)} = \nabla_0^{-1} p_2^{(1)}. \quad (3.25)$$

Higher orders can be read off order by order from the equations

$$\begin{aligned} \sum_{m=2}^{k+1} \nabla_0 p_1^{(m)} &= \sum_{m=2}^{k+1} p_2^{(m)} \\ \sum_{m=2}^{k+1} \nabla_0 p_2^{(m)} &= -\frac{i}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{f^{(n)}(p_1^{(0)})}{n!} \left(\sum_{m=1}^k p_1^{(m)} \right)^n (\chi_1^* \overleftrightarrow{\partial}_0 \chi_1) \\ &= -\frac{i}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{f^{(n)}(p_1^{(0)})}{n!} \sum_{l_1=0}^n \sum_{l_2=0}^{l_1} \cdots \sum_{l_{k-2}=0}^{l_{k-1}} \binom{n}{l_1} \binom{l_1}{l_2} \cdots \binom{l_{k-2}}{l_{k-1}} \\ &\quad \cdot \underbrace{\left(p_1^{(1)} \right)^{n-l_1} \left(p_1^{(2)} \right)^{l_1-l_2} \cdots \left(p_1^{(k-1)} \right)^{l_{k-2}-l_{k-1}} \left(p_1^{(k)} \right)^{l_{k-1}}}_{\text{of order } n+l_1+l_2+\cdots+l_{k-1}+1} \cdot (\chi_1^* \overleftrightarrow{\partial}_0 \chi_1). \end{aligned} \quad (3.26)$$

For a given order m this yields restrictions

$$m = n + l_1 + l_2 + \cdots + l_{k-1} + 1, \quad n \geq l_1 \geq l_2 \geq \cdots \geq l_{k-1} \geq 0 \quad (3.28)$$

which can be used to pick the terms of order m on the right hand side of eq. (3.27). For a given m the highest order in p_1 that can contribute to the equation determining $p_2^{(m)}$ is contained in the term proportional to

$$f^{(1)}(p_1^{(0)})p_1^{(m-1)}(\chi_1^* \overleftrightarrow{\partial}_0 \chi_1),$$

corresponding to $n = l_1 = \dots = l_{m-2} = 1, l_{m-3} = \dots = l_{k-1} = 0$. Thus the m th order $p_{1/2}^{(m)}$ only depend on the previous orders, and this expansion allows to solve eqs. (3.12) and (3.13) order by order in the fermion bilinears. The $(k+1)$ th order can also be calculated from eqs. (3.26) and (3.27), and at the end the truncation of eq. (3.21) turns out to be a technical assumption needed for applicability of the binomial theorem and, of course, for avoiding convergence issues of this expansion. Nonetheless the right hand side of eq. (3.27) contains all the terms necessary to calculate $p_{1/2}$ to arbitrary high orders.

Eq. (3.14) can always be solved, yielding (for $j_1 = 0$ and with eq. (3.20))

$$p_3 = \hat{B}_3 = e^{-Q(\hat{B}_1)} \left[\nabla_0^{-1} e^{Q(\hat{B}_1)} \left(j_3 - V(\hat{B}_1) + h(\hat{B}_1) g(\overline{\chi} \chi) + \frac{i}{\sqrt{2}} f(\hat{B}_1) (\chi_0^* \overleftrightarrow{\partial}_0 \chi_0) \right) + \tilde{p}_3 \right]. \quad (3.29)$$

Although the treatment of general dilaton couplings is possible, the physically most interesting cases of Einstein-Hilbert gravity with in four dimensions minimally coupled scalars, fermions and Yang-Mills fields yield upon spherical reduction kinetic terms with minimal or linear dilaton coupling [87], so that these two cases seem to be the most important ones. This is easily understood: Because the spherical symmetric Ansatz

$$(ds)^2 = h_{\alpha\beta} dx^\alpha dx^\beta - X(x^\alpha) (\sin^2 \theta d\phi^2 + d\theta^2) \quad (3.30)$$

reduces the four-dimensional volume element $d^4x \sqrt{-g}$ upon integration over the angular part to $\propto d^2x X(x^\alpha) \sqrt{-h}$, spherical reduction of in four dimensions minimally coupled matter will in general lead to a linear dilaton coupling in two dimensions. One important exception could be 2D Type 0A/0B string theory (cf. the penultimate model in table 2.1), if it could be generalized to include the same nonperturbative corrections that are already included in the exact string black hole (ESBH) [59, 60]. For the ESBH the first order gravity dilaton field X and the string dilaton $e^{-2\phi} = \gamma B$, with B being an auxiliary field, are related by

$$X = \gamma + \operatorname{arcsinh} \gamma. \quad (3.31)$$

The question now stands whether the string tachyon nonperturbatively couples to the first order gravity dilaton X , as it does in the perturbative limit (cf. §3.1 of [60]), or to the string dilaton γ . In the first case the coupling is linear, but in the second case it gets nonlinear corrections. Also some "two dilaton theories" [88] show nonlinear coupling to matter fields.

3.4 Ambiguous Terms

Eq. (3.15) as it stands is not the whole effective action, but contains an ambiguity [18, 89] arising from the source terms $J^i \hat{B}_i$. In expressions like $\int J \nabla^{-1} A$ the regularized inverse derivative, which stands for an integral operator, acts after a partial integration and changing the order of integration on the source J and in this way giving rise to another homogeneous contribution $\int \tilde{g} A$, whereas the homogeneous functions in A have already been made explicit in the solutions \hat{B}_i . The action thus has to be supplemented by three terms

$$\begin{aligned} \mathcal{L}_{amb} = & \sum_{i=1}^2 \tilde{g}_i K_i (\nabla_0^{-1}, (\chi_1^* \overleftrightarrow{\partial}_0 \chi_1), j_1, j_2) \\ & + \tilde{g}_3 e^{\hat{Q}} \left(j_3 - V(\hat{B}_1) + h(\hat{B}_1) g(\overline{\chi} \chi) + f(\hat{B}_1) \frac{i}{\sqrt{2}} (\chi_0^* \overleftrightarrow{\partial}_0 \chi_0) \right). \end{aligned} \quad (3.32)$$

The quantities K_i can be read off from the solutions $\hat{B}_{1/2}$ up to the desired order in matter contributions. The first two terms in general do not generate any new couplings of the matter fields besides the ones generated by the $-\tilde{g}_3 w'(\hat{B}_1) = -\tilde{g}_3 e^{\hat{Q}(\hat{B}_1, \hat{B}_2)} V(\hat{B}_1)$ and thus can be omitted unless $w'(X) = \text{const}$. In that case they become important and, for example, yield corrections to the specific heat of the CGHS black hole (cf. the third model in table 2.1) when coupled to a scalar field [86], changing it from a meaningless infinity to a positive finite value.

That these ambiguous terms are necessary and cannot be omitted is also obvious from (3.15). We know that classically the fermions couple to the metric determinant $\sqrt{-g} = q^3$ and this should still be the case after integration of the geometric degrees of freedom, where the coupling is to an effective background defined by the j_i and the matter fields. After setting $J^i = 0$ the action (3.15) is independent of j_3 and thus because of (3.10) containing a factor F^{-2} and the delta functional therein setting $F = 0$ the whole partition function would be ill-defined without the ambiguous terms.

Furthermore, first order gravity in two dimensions without matter is locally quantum trivial [16], i.e. the effective action expressed in terms of the mean

fields $\langle q^i \rangle = \delta S_{\text{eff}} / \delta j_i$ and $\langle p_i \rangle = \delta S_{\text{eff}} / \delta J^i$ is, up to a boundary term, the classical action (2.3) in Eddington-Finkelstein gauge. Thus no local quantum corrections appear in first order gravity and all eventual quantum effects are encoded in the boundary part and are of global nature. The mean fields then fulfill the classical equations of motion. In the matterless case, i.e. dropping the $\mathcal{D}\bar{\chi}\mathcal{D}\chi$ -integration and the covariant measure factor and setting $f(X) = h(X) = 0$, the correlator

$$\langle e_1^+ \rangle = \frac{1}{iZ[J=j=0]} \frac{\delta}{\delta j_3} e^{i \int d^2x (\mathcal{L}_{\text{eff}}^2 + \mathcal{L}_{\text{amb}})} \Big|_{j_i=J^i=0} = \tilde{g}_3 e^{Q(B_1[j_i=J^i=0])} \quad (3.33)$$

is indeed just the classical solution for e_1^+ , cf. eq. (2.24), in the gauge (2.107) (with $dx^1 = X^+ df$ and $\tilde{g}_3 = 1$). Thus for obtaining the right classical solutions in the theory without matter the third term in (3.32) is indispensable. This example also shows that the homogeneous functions \tilde{g}_i are fixed by imposing asymptotic conditions on the expectation values of the geometric fields q^i .

The occurrence of the ambiguity is also connected to the unusual order of integrating first over the canonical coordinates q^i and then over the momenta p_i . As shown in [90], for the special case of the Katanaev-Volovich model (the fourteenth entry in table 2.1) without matter the path integration can be carried out in "normal" order, i.e. first over p_i and then over q^i , without introducing sources J^i , yielding an effective action exactly of the type of the last term of eq. (3.32).

After quantizing the whole ghost and geometry sector, the generating functional reads

$$Z[J, j, \eta, \bar{\eta}] = \mathcal{N} \int \mathcal{D}F \delta \left(F - \frac{1}{i} \frac{\delta}{\delta j_3} \right) \tilde{Z}[F, J, j, \eta, \bar{\eta}] \quad (3.34)$$

$$\tilde{Z}[F, J, j, \eta, \bar{\eta}] = \int \mathcal{D}\bar{\chi} \mathcal{D}\chi e^{i \int d^2x (\mathcal{L}_{\text{eff}}^2 + \mathcal{L}_{\text{amb}})} \Big|_{\chi=F^{-\frac{1}{2}}\bar{\chi}}, \quad (3.35)$$

with $\mathcal{L}_{\text{eff}}^{(2)}$ from (3.15) and \mathcal{L}_{amb} from (3.32). It should be emphasised that Z includes all gravitational backreactions, because the auxiliary field F upon integration is equivalent to the quantum version of e_1^+ .

3.5 Conformal Properties of the Effective Action

The action of conformal transformation $g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}$ (or equivalently $g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu}$) after fixing the gauge and specifying asymptotic values of the p_i is

slightly non-trivial and requires some discussion. This is a necessary prerequisite to understand the conformal properties of the vertices and the scattering matrix. By assumption it will be required that the asymptotic values \tilde{p}_i and the gauge fixing conditions (2.107) are invariant. This implies in particular that neither e_0^- nor X^+ transform and that e_1^+ has the same conformal weight as the metric. Furthermore, the geometric part of the conserved quantity eq. (2.28) is required to be conformally invariant. This leads to different conformal weights (listed in table 3.1) as compared to the situation before gauge fixing, eq. (2.33). The conformal weight $w(A)$ of a field monomial A is defined (if it transforms homogeneously) by $A = \Omega^{w(A)} \tilde{A}$. Conformal weights thus add for products of field monomials, $w(AB) = w(A) + w(B)$.

Weight 2	Weight 1	Weight 0	Weight -1	Weight -2
$g_{\mu\nu}, e_1^+, J^3$	η_1	$e_0^\pm, e_1^-, X, X^+, \tilde{p}_i, \tilde{g}_i, \chi_1, J^{1/2}, j_{1/2}, \eta_0, A^+$	χ_0	X^-, j_3, A^-

Table 3.1: Conformal weights for Eddington-Finkelstein gauge

The gauge fixed spin-connection transforms inhomogeneously,

$$\omega_0 \rightarrow \tilde{\omega}_0 = \omega_0 = 0, \quad \omega_1 \rightarrow \tilde{\omega}_1 = \omega_1 + (X^+ e_1^- + X^- e_1^+) \frac{d \ln \Omega}{dX}, \quad (3.36)$$

and the dilaton potentials transform as in the gauge invariant theory, cf. sec. 2.1.3. With this choice the effective action (3.15) and (3.32) is conformally invariant at tree-level in the matter fields, i.e. before perturbatively performing the path integration over χ and thus taking into account quantum corrections from the fermions.

Chapter 4

Perturbative Treatment of the Matter Fields

After having obtained the effective action (3.15) and (3.32) the remaining matter integration in (3.35) is carried out perturbatively in this chapter, following the treatment of the scalar case [18]. One first splits the effective action (3.15) and (3.32) into terms independent of the fermions, in those quadratic in the spinor components and in higher order terms summarized in an interaction Lagrangian \mathcal{L}_{int} . The solutions of eqs. (3.12) and (3.13) up to quadratic fermion terms are given by the zeroth order solutions B_i (eqs. (3.16) and (3.17) with $\kappa = 0$) plus the first order contribution eqs. (3.25). The expansion (3.21) then reads

$$\hat{B}_1 = B_1 - \frac{i}{\sqrt{2}} \nabla_0^{-2} (f(B_1) (\chi_1^* \overleftrightarrow{\partial}_0 \chi_1)) + \mathcal{O}(\chi^4) \quad (4.1)$$

$$\hat{B}_2 = B_2 - \frac{i}{\sqrt{2}} \nabla_0^{-1} (f(B_1) (\chi_1^* \overleftrightarrow{\partial}_0 \chi_1)) + \mathcal{O}(\chi^4). \quad (4.2)$$

Expanding¹ eq. (3.19)

$$\hat{Q}(\hat{B}_1, \hat{B}_2) = Q(B_1, B_2) - \frac{i}{\sqrt{2}} \int_y G_{xy} f(B_{1y}) (\chi_1^* \overleftrightarrow{\partial}_0 \chi_1)_y + \mathcal{O}(\chi^4) \quad (4.3)$$

$$G_{xy} = \int_z \nabla_{0xz}^{-1} [U'_z B_{2z} \nabla_{0zy}^{-2} + U_z \nabla_{0zy}^{-1}] \quad (4.4)$$

¹The space-time points where the functions are evaluated at are denoted in subscript. Double subscripts like G_{xy} mean $G(x, y)$, and $\int_x = \int d^2x$.

and \hat{B}_3 (eq. (3.29)) up to quadratic terms in the fermions yields

$$\begin{aligned}\hat{B}_3 = & B_3 + \frac{i}{\sqrt{2}} \int_y H_{xy} f(B_{1y}) (\chi_1^* \overleftrightarrow{\partial}_0 \chi_1)_y \\ & + e^{-Q_x} \nabla_0^{-1} \left(e^Q \left(\frac{i}{\sqrt{2}} f(B_1) (\chi_0^* \overleftrightarrow{\partial}_0 \chi_0) + h(B_1) m \bar{\chi} \chi \right) \right)\end{aligned}\quad (4.5)$$

$$\begin{aligned}H(x, y) = & e^{-Q_x} \int_z \nabla_{0xz}^{-1} e^{Q_z} \{ [G_{xy} - G_{zy}] (j_3 - V)_z \\ & + V'_z \nabla_{0zy}^{-2} \} + \tilde{p}_{3x} e^{-Q_x} G_{xy}.\end{aligned}\quad (4.6)$$

A similar expansion of the ambiguous terms eq. (3.32) with $\tilde{g}_1 = \tilde{g}_2 = 0$ yields

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}^{(0)} + \mathcal{L}_{\text{eff}}^{(2)} + \mathcal{L}_{\text{int}} \quad (4.7)$$

$$\mathcal{L}_{\text{eff}}^{(0)} = J^i B_i + \tilde{g}_3 e^Q (j_3 - V(B_1)) \quad (4.8)$$

$$\begin{aligned}\mathcal{L}_{\text{eff}}^{(2)} = & \frac{i}{\sqrt{2}} f(B_1) \left[(\chi_1^* \overleftrightarrow{\partial}_1 \chi_1) - E_1^- (\chi_1^* \overleftrightarrow{\partial}_0 \chi_1) + F^{(0)} (\chi_0^* \overleftrightarrow{\partial}_0 \chi_0) \right] \\ & + F^{(0)} h(B_1) m \bar{\chi} \chi + \bar{\eta} \chi + \bar{\chi} \eta\end{aligned}\quad (4.9)$$

$$\begin{aligned}E_1^-(x) = & \int_y \left[J_y^1 \nabla_{0yx}^{-2} + J_y^2 \nabla_{0yx}^{-1} - J_y^3 H_{yx} \right. \\ & \left. + \tilde{g}_{3y} e^{Q_y} (G_{yx} (j_3 - V)_y - V'_y \nabla_{0yx}^{-2}) \right]\end{aligned}\quad (4.10)$$

$$E_1^{+(0)}(x) = e^{Q_x} \left[\int_y J_y^3 e^{-Q_y} \nabla_{0yx}^{-1} + \tilde{g}_{3x} \right] =: F^{(0)}.\quad (4.11)$$

One recognizes the kinetic term (2.50) of fermions on a curved background in Eddington-Finkelstein gauge (2.107) with a background metric

$$g_{\mu\nu} = F^{(0)} \begin{pmatrix} 0 & 1 \\ 1 & 2E_1^- \end{pmatrix}_{\mu\nu}.\quad (4.12)$$

This background is still the classical one, solely depending on sources for the geometric variables (q^i, p_i) and the zeroth order solutions B_i . If taking into account the first two ambiguous terms in (3.32), $\mathcal{L}_{\text{eff}}^{(0)}$ and E_1^- acquire additional contributions

$$\mathcal{L}_{\text{eff}}^{(0)} + \tilde{g}_1 (j_1 + B_2) + \tilde{g}_2 j_2 \quad (4.13)$$

and

$$E_1^-(x) + \tilde{g}_2(x) + \int_y \tilde{g}_{1y} \nabla_{0yx}^{-1}.\quad (4.14)$$

After redefining the interaction part of the Lagrangian density such that the background (4.12) depends on the full $E_1^+(x) = F(x) = \frac{\delta}{\delta j_3(x)} \int d^2z \mathcal{L}_{\text{eff}}(z)$ instead of its matter-independent part $F^{(0)}$, i.e. taking into account backreactions onto the metric determinant to all orders in the fermion fields, the generating functional (3.35) becomes

$$\begin{aligned} \tilde{Z} = & \exp \left(i \int d^2x \mathcal{L}_{\text{eff}}^{(0)} + \mathcal{L}_{\text{int}} \left[\frac{1}{i} F^{-\frac{1}{2}} \frac{\delta^L}{\delta \bar{\eta}} \right] \right) \times \\ & \times \int \mathcal{D}\bar{\chi} \mathcal{D}\tilde{\chi} \exp \left(i \int d^2x \frac{i}{2} f(B_1) \epsilon_{ab} \tilde{\epsilon}^{\nu\mu} E_\mu^b (\bar{\chi} \gamma^a \overleftrightarrow{\partial}_\mu \tilde{\chi}) + \bar{\eta} \tilde{\chi} + \bar{\chi} \tilde{\eta} \right). \end{aligned} \quad (4.15)$$

There is one problem to address in connection with this interpretation of fermions on an effective background: The metric becomes complex if one chooses the regularisation of the inverse derivatives as in [18]. To retain reality of the effective metric one could define it to be the solution of the equations of motion of (3.5), i.e. eqs. (3.12) - (3.14), the equations for q^i and the constraints $G_i = 0$, together with appropriate boundary conditions and for given external sources. In what follows the effective background is assumed to be real. When calculating vertices in sec. 4.2 below, instead of expanding the effective action a method (first introduced in [18]) that uses exactly this idea is applied. Another solution would be to find a regularisation which gives a real integral kernel ∇_0^{-1} .

4.1 One-Loop Effects and Bosonization

In this section the fermions are assumed to be massless and minimally coupled. A self interaction term $g(\bar{\chi}\chi) = \lambda(\bar{\chi}\chi)^2$ can be rewritten by introducing an auxiliary vector potential,

$$S_{SI}[\chi, g, A] = \frac{\lambda}{2} \int d^2x (F A_a^2 + 2A_a \bar{\chi} \gamma^a \tilde{\chi}), \quad (4.16)$$

which is integrated over in the path integral. The last term will be absorbed into the Dirac operator, resembling minimal coupling of fermions to the vector potential. After replacing ∂_μ in (4.15) with the covariant derivative ∇_μ containing the metric compatible and torsion-free spin connection (which drops out in two dimensions anyway), partially integrating the kinetic term in (4.15)², completing the square and evaluating the Gaussian integral over

²Or in other words using the fact that the Dirac operator \not{D} in Minkowski space is self-adjoint with respect to the Dirac inner product with $\bar{\chi} = \chi^\dagger \gamma^0$.

χ and $\bar{\chi}$ yields

$$\begin{aligned} \tilde{Z} = \exp \left(i \int d^2x \mathcal{L}_{\text{eff}}^{(0)} + \mathcal{L}_{\text{int}} \left[-iF^{-\frac{1}{2}} \frac{\delta^L}{\delta \bar{\eta}} \right] \right) \times \\ \times \int \mathcal{D}A \text{Det} \not{D} \exp \left(i \int d^2x \left(\frac{\lambda}{2} F A_a^2 - \bar{\eta} \not{D}^{-1} \tilde{\eta} \right) \right) \end{aligned} \quad (4.17)$$

with the Dirac operator

$$\not{D} = iE_a^\mu \gamma^a (\nabla_\mu - i\lambda A_\mu), \quad \nabla_\mu = \partial_\mu - \frac{1}{2} \omega_\mu \gamma_*. \quad (4.18)$$

Conformal transformations are homogeneous transformations of the metric $g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu}$. The Dirac operator without vector potential transforms homogeneously with weight $-3/2$ when acting on a spinor which itself has weight $-1/2$, i.e. $\not{D}[A=0]\chi \mapsto \Omega^{-\frac{3}{2}} \not{D}[A=0]\chi$, and the kinetic term

$$(\chi, \not{D}\chi) = \int d^2x \sqrt{-g} \bar{\chi} \not{D}\chi$$

is conformally invariant. To retain this property in the case of non-vanishing vector potential, A_a needs to have conformal weight -1 (coming from E_a^μ), while A_μ has weight 0 . With Eddington-Finkelstein gauge (2.107) fixed the appropriate weights are 0 for A^+ and -2 for A^- and A^μ .

The determinant of the Dirac operator is most easily calculated using heat kernel methods [91] in Euclidean space. There \not{D} is essentially self-adjoint [92] with respect to the ordinary inner product on spinor space

$$(\chi, \psi) = \int d^2x \sqrt{-g} \chi^\dagger \psi.$$

The square of the Dirac operator is then of Laplace type (cf. eqs. (3.27) in [91] with $A_\mu^5 = 0$),

$$D = \not{D}^2 = -(g^{\mu\nu} \nabla_\mu \nabla_\nu + E), \quad (4.19)$$

with

$$E = -\frac{R}{4} + \frac{i\lambda}{2} \gamma_* \epsilon^{\mu\nu} F_{\mu\nu} \quad (4.20)$$

and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ denotes the electromagnetic field strength. Conventions applied for Euclidean space can be found in App. A.

4.1.1 Zeta Function Regularization and the Heat Kernel

For deriving one-loop results for classically conformal and chiral invariant field theories the application of heat kernel methods and zeta function regularization [91] is most convenient. The goal of this subsection is to calculate the one-loop effective action

$$W_{\text{1loop}} = -\ln \text{Det} \not{D} = -\frac{1}{2} \ln \text{Det} D = -\frac{1}{2} \text{Tr}_{L^2}(\ln D). \quad (4.21)$$

For positive eigenvalues λ of D the formal identity

$$\ln \lambda = - \int_0^\infty \frac{dt}{t} e^{-t\lambda} \quad (4.22)$$

can be used to express the effective action

$$W = \frac{1}{2} \int_0^\infty \frac{dt}{t} K(t, D) \quad (4.23)$$

in terms of the heat kernel

$$K(t, f, D) = \text{Tr}_{L^2}(f e^{-tD}), \quad K(t, D) = K(t, 1, D). \quad (4.24)$$

Eq. (4.22) is only true up to an infinite contribution that does not depend on λ , as can be seen from the Laplace transform (using eq. (4.32))

$$\int_0^\infty t^{s-1} e^{-t\lambda} dt = \mathcal{L}[t^{s-1}] = \frac{\Gamma[s]}{\lambda^s} = \frac{1}{s} - \gamma_E - \ln \lambda + \mathcal{O}(s). \quad (4.25)$$

Here γ_E is the Euler-Mascheroni constant. Extending (4.22) to zero or negative eigenvalues is only formally possible, because the integral does not converge at the upper limit. These eigenvalues correspond to infrared divergences that can be regularized by e.g. adding a mass to the Laplace type operator, $D \mapsto D + m^2$. The divergence at the lower limit corresponds to the ultraviolet range and has to be properly regularized by e.g. zeta function regularization, i.e. by using the Laplace transformed effective action (4.23) before taking the limit $s \rightarrow 0$ as the regularized effective action

$$W_s = \frac{1}{2} \tilde{\mu}^{2s} \int_0^\infty \frac{dt}{t^{1-s}} K(t, D), \quad s \geq 0. \quad (4.26)$$

The quantity $\tilde{\mu}$ is a constant with the dimension of a mass introduced to keep the dimension of the effective action unchanged. The regularization is removed in the limit $s \rightarrow 0_+$. The key result used below is that on manifolds of dimension n without boundaries (or on manifolds with boundaries and local Neumann, Dirichlet or mixed boundary conditions) the heat kernel (4.24) admits an asymptotic expansion for small t ,

$$K(t, f, D) \cong \sum_{k \geq 0} t^{(k-n)/2} a_k(f, D), \quad (4.27)$$

where the heat kernel coefficients a_k are locally computable from invariants of the manifold, of the bundle part E of the operator (4.19) and possibly more invariants if the smearing function f itself takes values in some internal gauge group. They are tabulated in e.g. [91]. The zeta function for a positive operator D is defined as

$$\zeta(s, f, D) = \text{Tr}_{L^2} (f D^{-s}), \quad \zeta(s, D) = \zeta(s, 1, D), \quad (4.28)$$

and is related to the heat kernel (4.24) by the Mellin transform

$$\zeta(s, f, D) = \Gamma(s)^{-1} \int_0^\infty dt t^{s-1} K(t, f, D) \quad (4.29)$$

$$K(t, f, D) = \frac{1}{2\pi i} \oint ds t^{-s} \Gamma(s) \zeta(s, f, D), \quad (4.30)$$

where the integration contour in the second equation encircles all poles of the integrand. For operators with negative and zero modes one should replace D in (4.28) with the absolute value $|D|$ and restrict the sum in the trace to non-zero eigenvalues. The zeta function is regular at $s = 0$. Comparing (4.27) with (4.30) relates the heat kernel coefficients to poles of the zeta function,

$$a_k(f, D) = \text{Res}_{s=(n-k)/2} (\Gamma(s) \zeta(s, f, D)), \quad (4.31)$$

which becomes clear after recalling that the gamma function has a simple pole at $s = 0$,

$$\Gamma(s) = \frac{1}{s} - \gamma_E + \mathcal{O}(s). \quad (4.32)$$

In particular

$$a_n(f, D) = \zeta(0, f, D) \quad (4.33)$$

will be used in the following. The regularized action (4.26) can now be expressed as

$$W_s = \frac{1}{2} \tilde{\mu}^{2s} \Gamma(s) \zeta(s, D), \quad (4.34)$$

and expanded around $s = 0$ with the use of (4.32) ($\mu^2 = e^{-\gamma_E} \tilde{\mu}^2$), revealing a pole at $s = 0$,

$$W_s = \frac{1}{2} \left(\frac{1}{s} + \ln \mu^2 \right) \zeta(0, D) + \frac{1}{2} \zeta'(0, D), \quad (4.35)$$

that has to be removed by renormalization such that the limit $s \rightarrow 0_+$ can be safely taken. The remaining part is the renormalized effective action

$$W^{\text{ren}} = \frac{1}{2} \zeta'(0, D) + \frac{1}{2} \ln(\mu^2) \zeta(0, D). \quad (4.36)$$

The variation of the zeta function can be rigorously derived [91] by use of the Mellin transform eqs. (4.29), (4.30) and coincides with the naive variation

$$\delta \zeta(s, D) = -s \text{Tr}_{L^2}((\delta D) D^{-s-1}). \quad (4.37)$$

4.1.2 Conformal Anomaly

The energy-momentum tensor in classical field theory with action S on a curved Euclidean background is defined as

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}. \quad (4.38)$$

If the theory is conformally invariant, the variation of the action has to vanish under arbitrary (infinitesimal) conformal transformations of the metric $\delta g^{\mu\nu} = -2\delta\rho g^{\mu\nu}$,

$$0 = \delta S = \frac{1}{2} \int d^2x \sqrt{g} T_{\mu\nu} \delta g^{\mu\nu} = - \int d^2x \sqrt{g} T_{\mu}^{\mu} \delta\rho, \quad (4.39)$$

and thus the energy-momentum tensor of a classically conformally invariant field theory is necessarily traceless,

$$T_{\mu}^{\mu} = T_{\mu\nu} g^{\mu\nu} = 0. \quad (4.40)$$

In the quantum theory this no longer needs to be the case. Defining the Euclidean quantum effective action as $W = -\ln Z$ and the expectation value of the energy-momentum tensor as

$$\langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{g}} \frac{\delta W}{\delta g^{\mu\nu}}, \quad (4.41)$$

the variation of the effective action is related to the trace of the quantum energy momentum tensor $T_{\mu}^{\mu} = g^{\mu\nu} \langle T_{\mu\nu} \rangle$ like in (4.39) with S replaced by

W . For the classical action to be conformally invariant, the square of the Dirac operator has to transform conformally covariant, $D \rightarrow e^{-2\rho} D$. The infinitesimal variation is $\delta D = -2(\delta\rho)D$, and varying the renormalized effective action (4.36) yields (cf. (4.37))

$$\begin{aligned}\delta W^{\text{ren}} &= \frac{1}{2} \frac{d}{ds} \Big|_{s=0} \delta\zeta(s, D) + \frac{1}{2} \ln(\mu^2) \delta\zeta(0, D) \\ &= \zeta(0, \delta\rho, D) \\ &= a_2(\delta\rho, D).\end{aligned}\tag{4.42}$$

Comparing with eq. (4.39) for the quantum effective action shows that the trace of the quantum energy-momentum tensor eq. (4.41) is proportional to the second local heat kernel coefficient whose structure [91] is especially simple, yielding with eq. (4.20) and $\text{tr}_{\mathbb{C}^2} \mathbf{1} = 2$

$$T_\mu^\mu = -a_2(x, D) = -\frac{1}{4\pi} \text{tr}_{\mathbb{C}^2} \left(\frac{R}{6} + E \right) = \frac{R}{24\pi}.\tag{4.43}$$

The field strength term in E does not contribute because of the spinor trace $\text{tr}_{\mathbb{C}^2}(\gamma_*) = 0$. Notably, the conformal anomaly for a Dirac fermion in two dimensions has the same value as for one real boson [91], and can immediately be used to integrate (with initial condition $W[g_{\mu\nu} = \eta_{\mu\nu}] = 0$) the variation of the 1-loop effective action (now in Minkowski space) to the nonlocal Polyakov action [93]

$$W_{\text{Pol}}[g] = \frac{1}{96\pi} \int_{\mathcal{M}} d^2x \sqrt{-g} R \frac{1}{\Delta} R,\tag{4.44}$$

with the Laplacian on functions ($d^\dagger f(x) = 0$, cf. (A.8))

$$\Delta f = d^\dagger d = * d * d = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu f),\tag{4.45}$$

and Green's function defined as

$$\Delta_x \Delta^{-1}(x, y) = \delta(x - y).\tag{4.46}$$

4.1.3 Chiral Anomaly

Although the whole action in eq. (4.17) is not $U(1)$ gauge invariant because of the coupling-dependent mass term in (4.17), the action in the path integral which yields the determinant of the Dirac operator,

$$\text{Det} \not{D} = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{i \int d^2x \bar{\psi} \not{D} \psi},\tag{4.47}$$

is invariant under $U(1)$ gauge transformations (both in Minkowski and Euclidean space)

$$\delta_\Lambda \psi = -i\Lambda\psi, \quad \delta_\Lambda \bar{\psi} = i\bar{\psi}\Lambda, \quad \delta_\Lambda A_\mu = -\lambda^{-1}(\partial_\mu \Lambda). \quad (4.48)$$

Here the vector potential A_μ is treated as a background field, and Λ is a real gauge potential. It is also invariant under global chiral transformations,

$$\tilde{\delta}_\varphi \psi = -i\varphi\gamma_*\psi, \quad \tilde{\delta}_\varphi \bar{\psi} = -i\bar{\psi}\varphi\gamma_*, \quad (\varphi \in \mathbb{R}). \quad (4.49)$$

In Euclidean space this transformation rule still applies, but with the conventions chosen in App. A γ_* becomes anti-hermitian, in contrast to the Minkowski case where it is hermitian. Thus in the Euclidean case a hermitian representation of the chiral symmetry group $U(1)$ rather than a unitary one is used. To gauge this chiral symmetry fermions have to be coupled to an axial vector A_μ^5 (the gauge potential for the chiral symmetry) transforming as

$$\tilde{\delta}_\varphi A_\mu^5 = \begin{cases} -\lambda^{-1}\partial_\mu \varphi & \text{(M)} \\ i\lambda^{-1}\partial_\mu \varphi & \text{(E)} \end{cases}, \quad (4.50)$$

for Minkowski (M) and Euclidean (E) space, respectively. The vector potential A_μ is invariant under chiral transformations, as is the axial vector under gauge transformations. The locally chiral and gauge invariant action then reads

$$S = (\psi, \not{D}\psi) = \int d^2x \sqrt{\mp g} \bar{\psi} \not{D}\psi \quad (4.51)$$

$$\not{D} = \begin{cases} iE_a^\mu \gamma^a (\nabla_\mu - i\lambda A_\mu - i\lambda \gamma_* A_\mu^5) & \text{(M)} \\ iE_a^\mu \gamma^a (\nabla_\mu - i\lambda A_\mu + \lambda \gamma_* A_\mu^5) & \text{(E)} \end{cases}, \quad (4.52)$$

with $\bar{\psi} = \psi^\dagger \gamma^0$ for Minkowski space and $\bar{\psi} = \psi^\dagger$ for Euclidean space. The Euclidean Dirac operator and its square transform as

$$\delta_\Lambda \not{D} = i[\not{D}, \Lambda], \quad \tilde{\delta}_\varphi \not{D} = i\{\varphi\gamma_*, \not{D}\}, \quad (4.53)$$

$$\delta_\Lambda D = i[D, \Lambda], \quad \tilde{\delta}_\varphi D = i\{\varphi\gamma_*, D\} + 2i\varphi \not{D}\gamma_* \not{D}. \quad (4.54)$$

Because of the vanishing variation eq. (4.37) of the zeta function,

$$\tilde{\delta}_\varphi \zeta(s, D) = -is \text{Tr}_{L^2} ([D, \Lambda] D^{-s-1}) = i \text{Tr}_{L^2} ([D^{-s}, \Lambda]) = 0, \quad (4.55)$$

$U(1)$ gauge invariance is retained on quantum level. The zeta function is not invariant under chiral transformations,

$$\tilde{\delta}_\varphi \zeta(s, D) = -4is \text{Tr}_{L^2} (\varphi\gamma_* D^{-s}) = -4is \zeta(s, \varphi\gamma_*, D), \quad (4.56)$$

but yields a non-vanishing chiral anomaly (for $A_\mu^5 = 0$)

$$\begin{aligned}
\mathcal{A}(\varphi) = \tilde{\delta}_\varphi W^{\text{ren}} &= -2i\zeta(0, \varphi\gamma_*, D) \\
&= -2ia_2(\varphi\gamma_*, D) \\
&= \frac{1}{2\pi i} \int d^2x \sqrt{g} \varphi \text{tr}_{\mathbb{C}^2} \left[\gamma_* \left(\frac{R}{6} + E \right) \right] \\
&= -\frac{\lambda}{2\pi} \int d^2x \sqrt{g} \varphi \epsilon^{\mu\nu} F_{\mu\nu} \\
&= -\frac{\lambda}{2\pi} \int d^2x \sqrt{g} \varphi * (dA). \tag{4.57}
\end{aligned}$$

Decomposing the one-form A uniquely into a gauge and an axial part

$$A = d\Lambda + d^\dagger * \varphi, \tag{4.58}$$

the field strength only depends on the axial part (with eqs. (A.4), (A.8)),

$$F = dA = dd^\dagger * \varphi = -d * d\varphi. \tag{4.59}$$

With eq. (4.45) the chiral anomaly becomes (now in Minkowski space)

$$\begin{aligned}
\tilde{\delta}_{\delta\varphi} W &= \frac{\lambda}{2\pi} \int d^2x \sqrt{-g} \delta\varphi \Delta\varphi \\
&= W_{\text{WZ}}[\varphi + \delta\varphi] - W_{\text{WZ}}[\varphi], \tag{4.60}
\end{aligned}$$

with W being the Wess-Zumino action [94]

$$W_{\text{WZ}} = \frac{\lambda}{4\pi} \int d^2x \sqrt{-g} (*F) \frac{1}{\Delta} (*F). \tag{4.61}$$

The full one-loop effective action thus comprises the Polyakov part eq. (4.44) and the Wess-Zumino part eq. (4.61),

$$W_{\text{1loop}} = -\ln \text{Det} \not{D} = W_{\text{Pol}} + W_{\text{WZ}}. \tag{4.62}$$

What remains to be evaluated is the path integral over the vector potential in eq. (4.17), now with the highly non-local integrand eq. (4.61). The nature of the application should thus decide whether it is favorable to use this form, or to treat the Thirring term directly as an interaction vertex. On the other hand it may well be that the Wess-Zumino action becomes local in some gauges, as happens for the Polyakov action in conformal gauge. Otherwise

auxiliary scalar fields can be introduced to bring the Polyakov and Wess-Zumino action to local form,

$$W_{\text{Pol}} = \frac{1}{48\pi} \int d^2x \sqrt{-g} \left[\frac{1}{2} (\nabla \Phi)^2 + \Phi R \right] \quad (4.63)$$

$$W_{\text{WZ}} = \frac{\lambda}{2\pi} \int d^2x \sqrt{-g} \left[\frac{1}{2} (\nabla Y)^2 + Y(*F) \right]. \quad (4.64)$$

These expressions coincide with the ones in [32, 33].

Rewriting the Thirring term as in (4.16) is actually the application of the bosonization prescription found by Coleman, Jackiw and Susskind [34, 35] for the Schwinger model in flat Minkowski space. Transforming to a gauge with $\Lambda = 0$ in eq. (4.58) leads to the identification $\epsilon^{\mu\nu} \partial_\mu \varphi \propto \bar{\chi} \gamma^\mu \chi$. The treatment in this section thus shows that bosonization is also applicable within the framework of quantized dilaton gravity in two dimensions even after the geometric sector is quantized nonperturbatively.

4.2 Lowest Order Vertices

Because the fermions appear as bilinears in the effective action eqs. (3.15) and (3.32), the lowest order interaction vertices are non-local four-point vertices, and all vertices have an even number of outer legs. In the following only massless fermions without self interactions ($g(\bar{\chi}\chi) = 0$) are considered, thus the vertices are solely generated by gravitational interaction.

4.2.1 Effective Geometry and the Virtual Black Hole

The standard way for extracting the form of interaction vertices would be to expand the effective action eqs. (3.15) and (3.32) up to fourth order in the matter fields and read off the vertex as the corresponding coefficient. This is very cumbersome because of nested integrals coming from the inverse derivatives, whose ranges of integration have to be treated properly. We therefore adopt a strategy introduced in [18] and directly solve the equations of motion for the canonical pair (q^i, p_i) following from the action eq. (3.5) (together with the first order gravity constraints eqs. (2.68)-(2.70) and for vanishing sources $j_i = J^i = 0$),

$$\partial_0 p_1 = p_2 \quad (4.65)$$

$$\partial_0 p_2 = -2f(p_1)\Phi_0 \quad (4.66)$$

$$\partial_0 p_3 = -U(p_1)p_2p_3 - V(p_1) + 2f(p_1)\Phi_1 \quad (4.67)$$

$$\partial_0 q^1 = q^3(U'(p_1)p_2p_3 + V'(p_1)) + 2f'(p_1)[q^2\Phi_0 - q^3\Phi_1 - \Phi_2] \quad (4.68)$$

$$\partial_0 q^2 = -q^1 + q^3p_3U(p_1) \quad (4.69)$$

$$\partial_0 q^3 = q^3p_2U(p_1), \quad (4.70)$$

with matter contributions

$$\Phi_0 = \frac{i}{2\sqrt{2}}(\chi_1^* \overleftrightarrow{\partial}_0 \chi_1), \quad \Phi_1 = \frac{i}{2\sqrt{2}}(\chi_0^* \overleftrightarrow{\partial}_0 \chi_0), \quad \Phi_2 = \frac{i}{2\sqrt{2}}(\chi_1^* \overleftrightarrow{\partial}_1 \chi_1) \quad (4.71)$$

localized at a point y in space-time, i.e.

$$\Phi_i = c_i \delta^{(2)}(x - y), \quad i = 0, 1, 2. \quad (4.72)$$

Such a matter configuration is clearly off-shell, i.e. it does not fulfill the Dirac equation (4.102)-(4.103). Because of the structure of the interaction terms in the effective action eqs. (3.15) and (3.32), one can expect to find three four-point vertices

$$V^{(4)} = \int_x \int_z \left[V_a(x, z) \Phi_0(x) \Phi_0(z) + V_b(x, z) \Phi_0(x) \Phi_2(z) + V_c(x, z) \Phi_0(x) \Phi_1(z) \right].$$

After substituting the solutions of eqs. (4.65)-(4.70) back into the interaction terms of eq. (3.5) and expanding up to second order in the c_i the explicit form of the vertices is found. In order to be able to substitute the coefficients c_i by the corresponding bilinears (4.71) at the end, one should carefully keep track of the (because of non-locality of the interaction) different space-time points where the external legs are situated at.

The solutions of eq. (4.65)-(4.70) will have a delta peak in x^1 direction and the following continuity properties: p_2, p_3, q^1 will jump at y^0 because of the delta distributions on the right hand side, while the other variables will be continuous. To solve these equations unambiguously, a prescription for the behaviour of the vacuum solutions far away from the localization point is needed. To fix the six integration constants one requires [48] in the asymptotic region $x^0 > y^0$

- $p_1|_{x^0 > y^0} = x^0$ and $p_2|_{x^0 > y^0} = 1$, i.e. identification the dilaton with the x^0 -coordinate. This corresponds to the fundamental patch used in sec. 2.1.2, and can always be reached from the general solutions, eq. (4.73) and (4.74) below, by a residual gauge transformation eq. (2.115) with $\gamma(x^1) = -\ln c(x^1)$ and $g(x^1) = -d(x^1)/c(x^1)$ for $c(x^1) \neq 0$.
- $\mathcal{C}^{(g)}|_{x^0 > y^0} = \mathcal{C}_\infty$ fixes the integration constant in p_3 (see below).

- $q_3|_{x^0 > y^0} = e^{Q(x^0)}$ then solves eq. (4.70) and defines the asymptotic unit of length.
- The remaining two integration constants enter $q^2|_{x^0 > y^0}$, which is the solution of a second order partial differential equation derived below, are called m_∞ and a_∞ because for spherical reduced gravity they correspond to the Schwarzschild mass and the Rindler acceleration of the asymptotic background.

The general solutions for eqs. (4.65)-(4.67) and eq. (4.70) in the vacuum regions $x^0 \neq y^0$ are

$$p_1 = c(x^1)x^0 + d(x^1) \quad (4.73)$$

$$p_2 = c(x^1) \quad (4.74)$$

$$p_3 = e^{-Q(p_1)} \left[C(x^1) - \frac{w(p_1)}{p_2} \right] \quad (4.75)$$

$$q^3 = e(x^1)e^{Q(p_1)}. \quad (4.76)$$

For finding the vertices the solution for q^1 will not be necessary and, as only the exterior derivative of the spin connection enters the Ricci scalar

$$R = 2 * d\omega = -\frac{2}{q^3} \partial_0 q^1, \quad (4.77)$$

the scalar curvature can be read off directly from the right hand side of eq. (4.68). Another differentiation of eq. (4.69) with respect to x^0 and use of the other equations of motion yields a second order differential equation for q^2 ,

$$\partial_0^2 q^2 = -e(x^1)w''(p_1) - 2f'(p_1)[q^2\Phi_0 - q^3\Phi_1 - \Phi_2] + 2f(p_1)q^3U(p_1)\Phi_1, \quad (4.78)$$

which is solved in the vacuum regions by

$$q^2 = m(x^1) + a(x^1)p_1 - e(x^1)\frac{w(p_1)}{p_2^2}. \quad (4.79)$$

Adjusting the integration constants in the region $x^0 > y^0$ as listed above and patching the solutions at $x^0 = y^0$ according to their continuity properties (p_2 , p_3 , $\partial_0 q^2$ jumping, p_1 , q^3 , q^2 continuous) then fixes the integration constants

in the region $x^0 < y^0$, yielding up to linear order in the c^i

$$p_1 = x^0 + 2f(y^0)(x^0 - y^0)h_0 \quad (4.80)$$

$$p_2 = 1 + 2f(y^0)h_0 \quad (4.81)$$

$$p_3 = e^{-Q(p_1)} \left[\mathcal{C}_\infty - w(p_1) + 2f(y^0)h_0(w(x^0) - w(y^0)) - 2f(y^0)e^{Q(y^0)}h_1 \right] \quad (4.82)$$

$$q^2 = m_\infty + a_\infty x^0 - w(p_1) + 2h_0 \left[2f(y^0)(w(x^0) - w(y^0)) + [(m_\infty + a_\infty y^0)f'(y^0) - (fw)'|_{y^0}](x^0 - y^0) - 2(fe^Q)'|_{y^0}(x^0 - y^0)h_1 - 2f'(y^0)(x^0 - y^0)h_2 \right] \quad (4.83)$$

$$q^3 = e^{Q(p_1)}, \quad (4.84)$$

with $h_i = c_i \theta(y^0 - x^0) \delta(x^1 - y^1)$. The three constants m_∞ , \mathcal{C}_∞ and a_∞ are not independent from each other because the solutions in the asymptotic region $x^0 > y^0$ have to fulfill the geometric secondary constraints eq. (2.68)-(2.70) to describe a consistent background. The Lorentz constraint eq. (2.68) then requires

$$\mathcal{C}_\infty = m_\infty, \quad a_\infty = 0. \quad (4.85)$$

Eq. (2.69) even yields a vacuum solution for the spin connection,

$$q^1 = \frac{q^3}{p_2} \mathcal{V}(p_1; p_2 p_3), \quad (4.86)$$

and eq. (2.70) is identically fulfilled. With these solutions the geometric part of the conserved quantity eq. (2.28)

$$\mathcal{C}^{(g)} = m_\infty(1 + 2f(y^0)h_0) - 2f(y^0)w(y^0)h_0 - 2f(y^0)e^{Q(y^0)}h_1 \quad (4.87)$$

and the scalar curvature eq. (4.77)

$$R = -2(U'(p_1)p_2 p_3 + V'(p_1)) - 4f'(p_1)e^{-Q(p_1)}[q^2\Phi_0 - q^3\Phi_1 - \Phi_2] \quad (4.88)$$

both jump at $x^0 = y^0$. For non-minimal coupling the Ricci scalar even has a delta-like singularity at the point y , whereas for minimal coupling and $U' = 0$ it becomes continuous. The effective line element

$$(ds)^2 = 2e^+ \otimes e^- = 2q^3 dx^1 \otimes [dx^0 + q^2 dx^1] \quad (4.89)$$

takes Eddington-Finkelstein form

$$(ds)^2 = 2drdu + K(r, u, r_0, u_0)du^2 \quad (4.90)$$

by introducing new coordinates $dr = bq^3(x^0)dx^0$, $b > 0$, $du = b^{-1}dx^1$ (b is a real scale factor). The quantity

$$\begin{aligned} K(r, u, r_0, u_0) = & K_\infty [1 + 2f(y^0)U(x^0)(x^0 - y^0)h_0] - 4b^2e^{Q(x^0)}(x^0 - y^0) \times \\ & \times \left[(f(y^0)(w'(x^0) + w'(y^0)) + f'(y^0)(w(y^0) - m_\infty))h_0 + (fe^Q)'|_{y^0}h_1 + f'(y^0)h_2 \right] \\ & + 8b^2e^{Q(x^0)}f(y^0)(w(x^0) - w(y^0))h_0 \quad (4.91) \end{aligned}$$

with

$$K_\infty = 2b^2e^{Q(x^0)}(m_\infty - w(x^0)) \quad (4.92)$$

is the analogue of the Killing norm of the vacuum solutions, K_∞ . It is continuous at $x^0 = y^0$, as is the whole line element. The Ricci scalar (4.88) following from the Hodge dual of the spin connection is however not the same as the Ricci scalar computed from the line element (4.90)-(4.92), because it depends on the (for $U(X) \neq 0$) non-vanishing torsion part of the spin connection (cf. eq. (2.14)). The “torsion free” Ricci scalar in the asymptotic region $x^0 > y^0$ reads

$$\tilde{R} = \partial_r^2 K_\infty(r) = 2(V'(p_1) + p_2 p_3 U'(p_1)) - 4e^{-Q(p_1)}w''(p_1) \quad (4.93)$$

and coincides in the absence of torsion ($U(X) = 0$) with (4.88).

For minimally coupled spherical reduced gravity (cf. the first model in table 2.1; $f(X) = \kappa$) with $\lambda = 1/2$, $b = 1/(2\lambda) = 1$ and $m_\infty = 0$ eq. (4.91) contains a Rindler term ($\propto r$) and a Schwarzschild term ($\propto 1/r$),

$$K(r, u, r_0, u_0) = K_\infty - 4\kappa h_0 - \frac{2M}{r} + ar \quad (4.94)$$

$$M(r, u, r_0, u_0) = -\frac{3}{2}\kappa r_0 h_0 + \frac{4\kappa}{r_0}h_1 \quad (4.95)$$

$$a(r, u, r_0, u_0) = +\frac{\kappa}{r_0}h_0 + \frac{8\kappa}{r_0^3}h_1. \quad (4.96)$$

The dilaton coupling function $f(y^0)$ enters (4.91) only evaluated at the localization point y and thus the structure of the effective line element (4.90) does not change, showing that the virtual black hole is a generic feature for both minimal and non-minimal coupling.

As the asymptotic region $x^0 \rightarrow \infty$ in this example is Minkowski space ($m_\infty = 0$, $K_\infty = 1$), the occurrence of such an effective geometry is due to the scattering process, and in particular the occurrence of the Schwarzschild term is interpreted as the formation of a virtual black hole as an intermediary scattering state [89].

Fig. 4.1 shows the Carter-Penrose diagram for the resulting space-time. The non-trivial geometry is located on a light-like cut $u = u_0$ extending from $r = 0$ up to a maximal radius $r \leq r_0$. The geometry is non-local in the sense that it depends on two space-time coordinates (r_0, u_0) and (r, u) . The rest of space-time is flat Minkowski space or, in the general case, the background given by the line element eq. (4.90) with the asymptotic Killing norm eq. (4.92). As a word of warning this intermediary geometry should not be over-interpreted. It is off-shell, as the localized matter contributions are, and it is as much virtual as virtual particles in ordinary perturbative quantum field theory are.

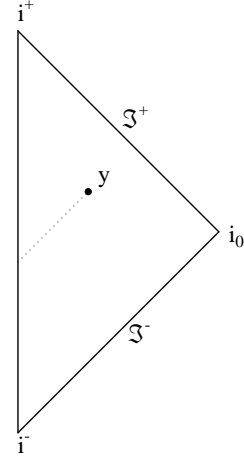


Figure 4.1: VBH

4.2.2 Four-Point Vertices

Inserting the solutions eqs. (4.80)-(4.84) and (4.85) into the interaction terms of eq. (3.5) (cf. also (4.71) and (4.72)),

$$S_{\text{int}} = -2 \int_x f(p_1) [q^2 \Phi_0 - q^3 \Phi_1 - \Phi_2], \quad (4.97)$$

expanding up to first order in the c^i , replacing these coefficients with $\Phi_i(y)$ and integrating over y yields three four-point vertices, namely the symmetric one

$$\begin{aligned} V_a = & -4 \int_x \int_y \Phi_0(x) \Phi_0(y) \theta(y^0 - x^0) \delta(x^1 - y^1) f(x^0) f(y^0) \times \\ & \times \left[2(w(x^0) - w(y^0)) - (x^0 - y^0)(w'(x^0) + w'(y^0)) \right. \\ & \left. - (x^0 - y^0) \left(\frac{f'(x^0)}{f(x^0)} (w(x^0) - m_\infty) + \frac{f'(y^0)}{f(y^0)} (w(y^0) - m_\infty) \right) \right] \end{aligned} \quad (4.98)$$

and two non-symmetric ones

$$V_b = -4 \int_x \int_y \Phi_0(x) \Phi_2(y) \delta(x^1 - y^1) |x^0 - y^0| f(x^0) f'(y^0). \quad (4.99)$$

$$V_c = -4 \int_x \int_y \Phi_0(x) \Phi_1(y) \delta(x^1 - y^1) |x^0 - y^0| f(x^0) (f e^Q)'|_{y^0} \quad (4.100)$$

Interestingly, the vertices V_a and V_b are the same as for a real scalar field [48], and only V_c is new. They share some properties with the scalar case, namely:

1. They are local in the coordinate x^1 , and non-local in x^0 .
2. They vanish in the local limit ($x^0 \rightarrow y^0$) and V_b vanishes for minimal coupling.
3. They respect a \mathbb{Z}_2 symmetry $f(X) \mapsto -f(X)$.
4. The symmetric vertex depends only on the conformally invariant combination $w(X)$ and the asymptotic value m_∞ of the geometric part of the conserved quantity (2.28). V_b is independent of U , V and m_∞ . Thus if m_∞ is fixed in all conformal frames, which is ensured by the conformal transformation properties listed in sec. 3.5, both vertices are conformally invariant.

The new vertex V_c is not conformally invariant because it contains $U(y^0)$, which is mapped to $\tilde{U}(y^0)$ under a change of the conformal frame. Still, conformal invariance of the four-fermi scattering matrix elements is retained at tree-level, because for on-shell matter configurations (cf. eqs. (4.111)-(4.113) below) the fermion bilinear Φ_1 and thus the vertex V_c vanishes. As all vertices containing Φ_1 are generated by the last term in eq. (3.32) it can be concluded that all tree-level Feynman diagrams containing pairs of outer χ_0 -legs attached to the same vertex must vanish, and all non-vanishing Feynman diagrams containing outer χ_0 -legs also have to contain internal χ_0 -propagators. Whether these diagrams also vanish or whether they yield conformally invariant S-matrix elements is not guaranteed *a priori*, but tree-level conformal invariance of the S-matrix is expected to hold because of the nonperturbative result obtained in sec. 3.5. At one-loop level, however, conformal invariance is broken by the conformal anomaly, cf. sec. 4.1.2. Thus as in the scalar case, where for minimal coupling the four-particle scattering amplitude for spherical reduced gravity (cf. the first model in table 2.1) turned out to be either zero or infinity [89] depending on which boundary conditions at the origin were chosen for the asymptotic states, only the symmetric vertex V_a will contribute at tree-level.

4.3 Asymptotic Matter States

The metric obtained from eq. (4.89) (and eqs. (4.83) and (4.84) with $a_\infty = 0$) in the region $x^0 \rightarrow \infty$ determines the asymptotic states for calculation of the scattering matrix. If they form a complete and (in an appropriate sense) normalizable set, an asymptotic Fock space can be constructed. In the Dirac equation obtained from eq. (2.50) with the Zweibein in Eddington-Finkelstein

gauge (2.107) and with

$$q^2(x^0) = m_\infty - w(x^0), \quad q^3(x^0) = e^{Q(x^0)}, \quad f(p_1) = f(x^0) \quad (4.101)$$

the spinor components decouple for massless fermions,

$$\partial_0 \chi_0 = -\frac{1}{2} \left(\frac{f'(x^0)}{f(x^0)} + U(x^0) \right) \chi_0 \quad (4.102)$$

$$(\partial_1 - q^2(x^0)\partial_0)\chi_1 = \frac{1}{2} \left(\frac{f'(x^0)}{f(x^0)} q^2(x^0) + q^{2'}(x^0) \right) \chi_1. \quad (4.103)$$

These equations are just the Dirac equation $\not{D}\chi = 0$ (cf. eq. (4.18)) in components, with a spin connection

$$\omega_0 = -U(x^0) - \frac{f'(x^0)}{f(x^0)} \quad (4.104)$$

$$\omega_1 = -U(x^0)q^2(x^0) - q^{2'}(x^0) - 2\frac{f'(x^0)}{f(x^0)}q^2(x^0). \quad (4.105)$$

The equation for χ_1 is conformally invariant (with conformal weights as in table 3.1), but the one for χ_0 contains $U(X)$ and thus transforms while changing between conformal frames. To solve the second equation one introduces new coordinates (C, C' are constants)

$$v(x^0, x^1) = x^1 - \int_{x^0}^{\infty} \frac{dz}{q^2(z)} + C \quad (4.106)$$

$$w(x^0, x^1) = x^1 + \int_{x^0}^{\infty} \frac{dz}{q^2(z)} + C' \quad (4.107)$$

in patches where the integrals are defined. In this way both coordinates are locally orthogonal to each other, $\partial_v w = \partial_w v = 0$. With the Ansatz

$$\chi_i = \mathcal{R}_i e^{i\phi_i}; \quad \mathcal{R}_i, \phi_i \text{ real} \quad (4.108)$$

one finds that \mathcal{R}_0 has to fulfill eq. (4.102), \mathcal{R}_1 has to obey

$$\partial_v \mathcal{R}_1 = \frac{1}{4} \left(\frac{f'(x^0)}{f(x^0)} q^2(x^0) + q^{2'}(x^0) \right) \mathcal{R}_1 \Big|_{x^0=x^0(v, x^1)}, \quad (4.109)$$

and the phases ϕ_i obey

$$\partial_0 \phi_0 = 0 \quad \text{and} \quad \partial_v \phi_1 = 0. \quad (4.110)$$

Most notably, solutions can be obtained explicitly even for general non-minimal coupling,

$$\chi_0(x^0, x^1) = \frac{F(x^1)}{\sqrt{f(x^0)}} \exp \left[i\phi_0(x^1) - \frac{Q(x^0)}{2} \right] \quad (4.111)$$

$$\begin{aligned} \chi_1(x^0, x^1) &= \tilde{F}(w) e^{i\phi_1(w)} \times \\ &\times \exp \left[\frac{1}{4} \int_{x^0=x^0(v', x^1)}^{v(x^0, x^1)} dv' \left(\frac{f'(x^0)}{f(x^0)} q^2(x^0) + q^{2'}(x^0) \right) \right] \end{aligned} \quad (4.112)$$

with $F(x^1)$ and $\tilde{F}(w)$ being arbitrary but real functions. The conformal transformation properties of these solutions are as expected in sec. 3.5. χ_0 includes a factor $e^{-Q(x^0)/2}$ and thus transforms with weight -1 , while χ_1 only depends on the conformally invariant combination $e^{Q(x^0)}V(x^0)$ and the mass m_∞ of the asymptotic space-time. Thus if m_∞ is fixed for all conformal frames, the solution for χ_1 is conformally invariant. The fermion bilinears (4.71) which enter the tree-level scattering vertices (e.g. the four-fermi vertices (4.98)-(4.99)) read on-shell

$$\Phi_0(x) = -\frac{|\chi_1|^2}{\sqrt{2}} \frac{\phi'_1(w)}{q^2(x^0)} = \frac{\Phi_2(x)}{q^2(x^0)}, \quad \Phi_1(x) = 0. \quad (4.113)$$

As noted at the end of the last section, the vanishing of Φ_1 implies conformal invariance of the tree-level scattering matrix elements for four-fermion scattering as well as vanishing of a certain class of tree-level Feynman diagrams.

Chapter 5

Conclusions and Possible Further Developments

5.1 Summary and Conclusions

The goal of this thesis was to analyze the classical structure and to quantize two-dimensional dilaton gravity models in the first order formulation coupled to Dirac fermions using the Feynman path integral approach. Chapter 1 contained a short motivation for considering gravity in two dimensions in general and first order gravity with fermions in particular, together with some historical remarks.

5.1.1 Classical Analysis

In chapter 2, after introducing the action describing the system under consideration, the equivalence between the first order formulation and the second order dilaton gravity action on the classical and quantum level was demonstrated, for which it was crucial that fermions in two dimensions do not directly couple to the spin connection and the Lagrange multipliers for torsion. After recalling how to obtain all classical solutions for the matterless theory or even for special matter configurations, namely (anti)chiral fermions and (anti)self-dual spinors, the constraints of the system were analyzed. Three primary first class constraints and four well-known primary second class constraints relating the spinor to its canonical momentum, as well as three secondary first class constraints generating local Lorentz transformations and infinitesimal diffeomorphisms were found. The Hamiltonian turned out to be fully constrained, as expected for a generally covariant theory.

The algebra of the secondary first class constraints, eqs. (2.76)-(2.79), is the main result of this analysis and should be compared to the known cases of first

order gravity without matter [16] and with a real scalar field [18, 19]. As for scalar matter, only the bracket between the two diffeomorphism generators (2.79) is changed in comparison to the matterless theory, but in the very specific way that the matter contribution is proportional to the local Lorentz generator. A new feature is that not only for minimal coupling but also for certain non-minimal couplings ($f \propto h$) the constraint algebra is the same as in the matterless case, namely if the fermions have non-vanishing mass but otherwise do not self-interact ($g(\bar{\chi}\chi) = m\bar{\chi}\chi$). Also the kinetic term of the fermions does not contribute at all, whereas in the scalar case [19] the additional structure function was just the kinetic term for the real boson. Furthermore, the algebra loses its property to be a finite W-algebra.

It is remarkable that the constraint algebra (2.76)-(2.79) closes without any spatial derivatives of delta functions and in this sense resembles rather an ordinary gauge theory or Ashtekar's approach to gravity [95, 96], while the linear combinations (2.89) correspond to the Hamiltonian (H_0) and diffeomorphism constraint (H_1) of the ADM approach [97] and fulfill the Virasoro algebra which closes with derivatives of delta functionals.

The treatment of boundaries for the matterless theory was also briefly discussed. It turns out that without any additional boundary terms in the first order gravity action (2.3) most of the constraint algebra is unchanged, but eq. (2.79) receives a boundary contribution (2.80) which vanishes for $(A)dS_2$ ground state models.¹

At the end of chapter 2 the BRST charge was constructed and Eddington-Finkelstein gauge (2.107) on the manifold and light cone gauge in tangent space was fixed. The homological perturbation theory terminates at Yang-Mills level, which can be inferred from the Poisson- σ model structure [11] of the matterless theory but is a non-trivial fact in the presence of matter when the theory does not admit a Poisson- σ model formulation. In general one would expect the presence of higher order ghost terms. The choice (2.107) does not fix the gauge completely but allows for residual gauge transformations (2.115), namely combined Lorentz transformations and diffeomorphisms.

5.1.2 Nonperturbative Quantization of Geometry

In chapter 3, the main part of this thesis, the path integral over the ghost sector and the geometric fields which were not fixed by the gauge choice (2.107) was performed nonperturbatively. I want to emphasize that this

¹Maybe one should further investigate whether there are any connections of this point to the topic of AdS_2/CFT_1 correspondence.

is only possible for two reasons. First, the constraints (2.65)-(2.67) which appear in the Hamiltonian density are linear in the fields $q^i = (\omega_1, e_1^-, e_1^+)$, and second, the gauge choice (2.107) not only further simplifies the structure of the Hamiltonian but also generates a Faddeev-Popov determinant (3.4) which solely depends on the canonical momenta $p_i = (X, X^+, X^-)$. The integration over q^i can then be carried out first, leading to the rather simple set of equations (3.12)-(3.14), and in the course of subsequently integrating over the p_i the Faddeev-Popov determinant then cancels.

Another remarkable point about this procedure is that the integration over the Zweibeine e_1^\pm is not restricted at all. In Eddington-Finkelstein gauge (2.107) the volume element is $\sqrt{-g} = e_1^+$, which classically should be restricted to positive values. But in order to carry out the nonperturbative path integration one has to integrate over the whole range of e_1^+ to arrive at the classical equations of motion for the p_i eqs. (3.12)-(3.14). Including quantum fluctuations of the metric corresponding to zero and negative volumes may thus also be the correct approach for quantizing other gravity theories, including general relativity.

Furthermore, this quantization of the geometric degrees of freedom is background independent in the sense that the metric has not been divided into a fiducial and a fluctuation part. Although the homogeneous parts in the solutions of eqs. (3.12)-(3.14) (the \tilde{p}_i) fix an asymptotic background at the infinitely far boundary, the geometry in the interior of the manifold is subjected to the quantum dynamics of the system and the quantization procedure does not depend on specific properties of special asymptotic backgrounds but is valid for all choices.

The resulting nonlocal and nonpolynomial effective action eqs. (3.15) and (3.32) depends on the spinors, external sources for the geometry and the asymptotic values \tilde{p}_i . It includes three ambiguous terms, which arise as a homogeneous solution of the (regularized) partial time derivative ∇_0 . They have to be present for several reasons discussed in sec. 3.4, namely ensuring quantum triviality of first order gravity without matter and for agreement with results obtained in cases where the more “natural” order of integrating first the momenta p_i and then the coordinates q^i (as e.g. in the Katanaev-Volovich model, the fourteenth entry in table 2.1) is possible. At the end of chapter 3 the resulting effective action was shown to be conformally invariant at tree-level in the matter fields.

5.1.3 Matter Perturbation Theory and Bosonization

Chapter 4 treats the remaining path integral over the Dirac fermions in the usual perturbative approach. The fermions were shown to propagate on

an effective background which consistently includes matter backreactions to arbitrary orders in the matter fields. At one-loop level both conformal and chiral symmetries develop anomalies, which were used to calculate the one-loop effective action consisting of a Polyakov and a Wess-Zumino part.

In that course it was shown that the quantum equivalence between bosons and fermions in flat two dimensional space-time found by Coleman, Jackiw and Susskind [34,35] can also be applied to the situation when dilaton gravity is quantized nonperturbatively. The prescription has been used, with the caveat that it only holds in regions where the space-time curvature is small compared to the microscopic length scale of the quantum theory, in [36–38] for semi-classical studies of the effects of pair production on the global structure of black hole space-times. It is now clear that this caveat can be dropped at the expense of the quantum theory becoming more complicated, with a path integration over the nonlocal Wess-Zumino action to be evaluated.

In the second part of this chapter the lowest order (i.e. four-particle) tree-level vertices were calculated for massless, not self-interacting fermions. They turn out to be nonlocal in the x^0 -coordinate and related to an intermediary off-shell geometry which can be thought of as a geometric state forming during the scattering process. For spherical reduced gravity (cf. the first model in table 2.1) the line element of this intermediary state contains a Schwarzschild and a Rindler term which is interpreted as the formation of a virtual black hole.

Though being a gauge-dependent feature it is noteworthy that two of the gravitationally induced four-fermion scattering vertices ((4.98) and (4.99)) found are the same as for a real, massless scalar field coupled to first order gravity [48]. These two vertices are invariant under conformal transformations. The third vertex, (4.100), although being new compared to the scalar case and not conformally invariant, was seen to vanish on-shell (cf. (4.113)), and thus for tree-level four-fermi scattering only the symmetric and non-symmetric vertices (4.98) and (4.99) contribute. The corresponding S-matrix elements are conformally invariant because the vertices V_a and V_b and the asymptotic states (4.112) are. Furthermore, all tree-level Feynman diagrams containing pairs of outer χ_0 -legs attached to the same vertex are found to vanish, such that scattering of χ_0 -particles into χ_1 -particles necessarily involves the exchange of virtual χ_0 -excitations, i.e. the presence of internal χ_0 -propagators in the corresponding Feynman diagrams.

Minimally coupled ($f(X) = \text{const.}$) models with $U(X) = 0$ and $V(X) = \text{const.}$ are found to be scattering trivial, i.e. all tree-level vertices vanish, as can be seen from eq. (3.15) and (3.32). This is a stronger requirement compared to the scalar case, where the last term in (3.32) was not present and where it thus was sufficient to require constancy of the $w' = e^Q V$ -term

in (3.32) (and minimal coupling). The CGHS black hole and Rindler ground state models (cf. table 2.1) could thus exhibit non-trivial fermion scattering and the class of scattering trivial models will be strongly restricted.

5.2 Possible Extensions

In [19] a number of interesting ideas have been listed for the case of first order gravity coupled to a real scalar field. Most of the remarks made there also apply to dilaton gravity with fermionic matter and will not all be repeated here. For other applications specific to fermions cf. also sec. 6.3. in [98].

As all needed tools (asymptotic states, vertices) for the evaluation of the scattering matrix elements for four-fermion scattering are provided in this thesis, such a calculation is the next natural step. For a massless real boson coupled to spherical reduced gravity (cf. the first model in table 2.1) the amplitude obtained in [19, 99] has been shown, among other things, to be unitary and the cross section to be CPT invariant [100, 101]. This is rather remarkable for two reasons: First, the effective theory obtained after quantizing geometry is non-local and thus the CPT theorem [102] does not apply. Secondly, as argued first in [103] and clarified later in [104], if the information paradox was really to hold and pure states could evolve into mixed ones through quantum gravity effects, then quantum mechanics itself should be modified to incorporate this non-unitary time evolution which in turn would lead to CPT violation. Reversely, the found CPT invariance of the S-matrix in the scalar case shows that the time evolution in these lowest order tree-level scattering processes is unitary. Investigation of CPT invariance in higher-order tree-level scattering and for loop calculations could thus shed some light on the information paradox. Also, on a real boson field charge conjugation acts trivially, while parity transformation is respected per constructionem in a spherical symmetric theory and time reversal can be achieved by changing from outgoing to ingoing Eddington-Finkelstein gauge. Hence investigating the case of Dirac fermions, on which C and P act nontrivial too, would be more interesting.

Besides that it is clear now how bosonization works on the level of the path integral (4.17), the fact that two of the vertices calculated in sec. 4.2 are exactly the same as for a free massless scalar field coupled to first order gravity [48] and that the contribution of the new vertex (4.100) vanishes for a large class of Feynman diagrams is another indication that bosonization also holds for the tree-level S-matrix elements. To clarify this issue the S-matrix elements for fermion scattering in some specific model (e.g. spherical reduced gravity) should be compared with the ones from the bosonic side of the

correspondence. Another remarkable fact is that the structure functions C_{ij}^k and in particular $C_{23}^1 = -h'(X)\lambda(\overline{\chi}\chi)^2$ of the constraint algebra (cf. (2.79) for couplings $f(X) = h(X)$ and $g(\overline{\chi}\chi) = m\overline{\chi}\chi + \lambda(\overline{\chi}\chi)^2$) is mapped to its counterpart in a free massless scalar theory [19] by the identification

$$S^\pm = ie^{\pm i\varphi} \sqrt{\lambda} \overline{\chi} \gamma^\pm \chi, \quad S^\pm = *(dS \wedge e^\pm), \quad (5.1)$$

with $S(x)$ being the scalar field and φ a phase. The Faddeev-Popov operators (2.110) of both theories are mapped onto each other, which however does not affect the quantum theories because the Faddeev-Popov determinant cancels during the exact path integral quantization of geometry in both cases. The constraints G_i themselves are however not identified under this mapping.

The treatment of boundaries in first order gravity coupled to matter fields is another open problem. As pointed out by Carlip [105–107], the universality of black hole entropy could be connected to the presence of a black hole horizon, which imposes constraints on the physical phase space, and an underlying symmetry somehow attached to the horizon. The last point seems to be connected to a conjecture by 't Hooft [68] that physical degrees of freedom should be converted into gauge degrees of freedom at a horizon. Indeed, in [60, 69] this was shown to be the case for first order gravity (2.3) without additional matter fields by constructing the reduced phase space. For the case of a generic boundary one physical degree of freedom was found, whereas for a horizon boundary the reduced phase space turned out to be empty. It would be interesting to see whether such a phenomenon also occurs in the presence of physical matter degrees of freedom. An indication in favor of such a phenomenon is the changing rank of the Dirac matrix (2.57) at a horizon $e_1^- = 0$. Boundary conditions for the matter fields can be introduced by means of imposing constraints at the boundary. On the technical side the Dirac consistency algorithm is more involved [108], with the Dirichlet boundary constraint for a scalar field generating an infinite tower of constraints similar to the flat space example considered in [109]. The Dirac matrix (2.57) then contains terms with distributional support at the boundary (i.e. $\delta(x_{\text{boundary}}^1)$ terms), so that a naive definition of the inverse matrix and construction of the Dirac bracket does not seem feasible. On the other hand, as opposed to the situation in flat space [109], on a curved space-time one can not impose the boundary conditions directly in Fourier space. A possible way out that could lead to a sensible constraint algebra allowing to judge which gauge symmetries survive in the presence of boundaries may be to use a regularization of the δ -functions at the boundary, define the Dirac bracket, calculate the constraint algebra and remove the regularization afterwards, but our efforts in that direction [108] so far have not been successful in removing all divergent contributions with support at the boundary.

Appendix A

Notational Conventions

Throughout this thesis natural units $\hbar = c = G_N = 1$ are used. Space-time is assumed to be a Lorentzian manifold with signature $(+ -)$. Latin indices a, b are local Lorentz indices, while Greek indices μ, ν refer to the manifold. The totally antisymmetric symbol both in tangent space and on the manifold is defined by $\epsilon_{01} = \tilde{\epsilon}_{01} = +1$, $\epsilon_{ab} = -\epsilon_{ba}$ and $\tilde{\epsilon}_{\mu\nu} = -\tilde{\epsilon}_{\nu\mu}$. The symbols with upper indices are defined as the usual matrix inverse. In light cone coordinates

$$u^\pm = \frac{1}{\sqrt{2}}(u^0 \pm u^1) \quad (\text{A.1})$$

it becomes $\epsilon^\pm_\pm = \pm 1$, where local Lorentz indices are raised with the flat metric $\eta_{ab} = \text{diag}(+-)$ and $\eta_{+-} = \eta_{-+} = 1$. Space-time indices are raised and lowered with the metric $g_{\mu\nu}$.

The components of a p-form are defined via

$$\Omega_p = \frac{1}{p!} \Omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}, \quad (\text{A.2})$$

and the Hodge dual of a p-form (in D dimensions) is defined with the coefficients

$$*\Omega_p = \Omega'_{D-p} = \frac{1}{p!(D-p)!} \epsilon_{\mu_1 \dots \mu_{D-p}}^{\nu_1 \dots \nu_p} \Omega_{\nu_1 \dots \nu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{D-p}}, \quad (\text{A.3})$$

with the Levi-Civita tensor $\epsilon_{\mu_1 \dots \mu_D} = |\det e_\mu^a| \tilde{\epsilon}_{\mu_1 \dots \mu_D}$. For even dimension D and Lorentzian signature this yields

$$**\Omega_p = (-1)^{p+1} \Omega_p \quad (\text{A.4})$$

and thus

$$*\epsilon = 1, \quad *1 = -\epsilon \quad (\text{A.5})$$

for the Hodge star acting on the volume form

$$\epsilon = (e) d^2 x \quad (\text{A.6})$$

$$(e) = \det(e_\mu^a) = e_0^+ e_1^- - e_1^+ e_0^- . \quad (\text{A.7})$$

With these conventions the hermitian conjugate of the exterior derivative [61] reads

$$d^\dagger = * d * . \quad (\text{A.8})$$

The volume form can be written as

$$\epsilon^{ab} \epsilon = e^a \wedge e^b \quad (\text{A.9})$$

$$\epsilon = -\frac{1}{2} \epsilon_{ab} e^a \wedge e^b = e^+ \wedge e^- . \quad (\text{A.10})$$

A Dirac fermion in two dimensions has two complex components, $\chi = (\chi_0, \chi_1)^T$. Dirac matrices in two-dimensional Minkowski space are chosen as

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \gamma^1 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \gamma^+ &= \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} & \gamma^- &= \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix} . \end{aligned} \quad (\text{A.11})$$

The analogue of the γ^5 matrix is $\gamma_* = \gamma_0 \gamma_1 = \text{diag}(+-)$. They satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ and $\{\gamma_*, \gamma^a\} = 0$. For calculations in Euclidean space γ^0 is defined as above, but $\gamma^1 = \text{diag}(+-)$ and $\gamma_* = \gamma_0 \gamma_1$, thus satisfying $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$. The Dirac matrices in Euclidean space are thus hermitian, $\gamma^a = \gamma^{a\dagger}$, whereas γ_* becomes anti-hermitian.

Appendix B

Details from Ch. 2

B.1 Poisson Brackets of the Secondary with the Second Class Constraints

To calculate the Dirac brackets, all Poisson brackets of the G_i with the Φ_α are needed. They are easily obtained by using the algebraic properties of the graded Poisson bracket, eqs. (2.45)-(2.46), and read

$$\begin{aligned}
\{G_1, \Phi'_0\} &= -\frac{i}{\sqrt{2}} f e_1^+ \chi_0^* \delta(x - x') \\
\{G_1, \Phi'_2\} &= -\frac{i}{\sqrt{2}} f e_1^+ \chi_0 \delta(x - x') \\
\{G_1, \Phi'_1\} &= -\frac{i}{\sqrt{2}} f e_1^- \chi_1^* \delta(x - x') \\
\{G_1, \Phi'_3\} &= -\frac{i}{\sqrt{2}} f e_1^- \chi_1 \delta(x - x') \\
\{G_2, \Phi'_0\} &= \frac{i}{\sqrt{2}} [f' + Uf] X^+ e_1^+ \chi_0^* \delta(x - x') \\
&\quad - e_1^+ h g' \chi_1^* \delta(x - x') \\
\{G_2, \Phi'_2\} &= \frac{i}{\sqrt{2}} [f' + Uf] X^+ e_1^+ \chi_0 \delta(x - x') \\
&\quad + e_1^+ h g' \chi_1 \delta(x - x').
\end{aligned}$$

A little care is needed when integrating by parts derivatives of the delta distributions. This is most easily done by smearing the fields with test functions and use of the identity

$$\int dx \varphi(x) [f(x) - f(y)] \partial_x \delta(x - y) = - \int dx \varphi(x) (\partial_x f(x)) \delta(x - y),$$

thus obtaining

$$\begin{aligned}
\{G_2, \Phi'_1\} &= \frac{i}{\sqrt{2}} [\chi_1^*(\omega_1 f - X^+ e_1^- f' - X^- e_1^+ U f) + 2(\partial_x \chi_1^*) f + (\partial_x f) \chi_1^*] \delta \\
&\quad - e_1^+ h g' \chi_0^* \delta \\
\{G_2, \Phi'_3\} &= \frac{i}{\sqrt{2}} [\chi_1(\omega_1 f - X^+ e_1^- f' - X^- e_1^+ U f) + 2(\partial_x \chi_1) f + (\partial_x f) \chi_1] \delta \\
&\quad + e_1^+ h g' \chi_0 \delta \\
\{G_3, \Phi'_0\} &= \frac{i}{\sqrt{2}} [\chi_0^*(\omega_1 f - X^- e_1^+ f' - X^+ e_1^- U f) - 2(\partial_x \chi_0^*) f - (\partial_x f) \chi_0^*] \delta \\
&\quad + e_1^- h g' \chi_1^* \delta \\
\{G_3, \Phi'_2\} &= \frac{i}{\sqrt{2}} [\chi_0(\omega_1 f - X^- e_1^+ f' - X^+ e_1^- U f) - 2(\partial_x \chi_0) f - (\partial_x f) \chi_0] \delta \\
&\quad - e_1^- h g' \chi_1 \delta \\
\{G_3, \Phi'_1\} &= \frac{i}{\sqrt{2}} [f' + U f] X^- e_1^- \chi_1^* \delta + e_1^- h g' \chi_0^* \delta \\
\{G_3, \Phi'_3\} &= \frac{i}{\sqrt{2}} [f' + U f] X^- e_1^- \chi_1 \delta - e_1^- h g' \chi_0 \delta.
\end{aligned}$$

Several technical points in the calculation of the constraint algebra may deserve some comments. Only obtaining (2.79) needs some care, the other brackets are calculated rather straightforward using the Poisson structure on phase space (2.43). The tricky part of (2.79) is actually not the $C^{\alpha\beta}$ -term in the Dirac bracket, but the bracket $\{G_2[\varphi], G_3[\psi]\}$ and therein the integrations by parts which have to be performed using smeared constraints

$$G_i[\varphi] = \int dx \varphi(x) G_i(x).$$

The bracket itself reads with (2.66), (2.67)

$$\begin{aligned}
\{G_2[\varphi], G_3[\psi]\} &= \iint dx dz \varphi(x) \psi(z) (\{G_2^{h=0}(x), G_3^{h=0}(z)\} \\
&\quad + \{q^3(x) h(x) g(x), G_3^{h=0}(z)\} - \{G_2^{h=0}(x), q^2(z) h(z) g(z)\}) . \quad (\text{B.1})
\end{aligned}$$

Here $G^{h=0}$ denote the constraints with $h = 0$,

$$\begin{aligned}
G_1^{h=0} &= G_1^g = G_1 \\
G_2^{h=0} &= G_2^g + \frac{i}{\sqrt{2}} f(X) (\chi_1^* \overleftrightarrow{\partial}_1 \chi_1) \\
G_3^{h=0} &= G_3^g - \frac{i}{\sqrt{2}} f(X) (\chi_0^* \overleftrightarrow{\partial}_1 \chi_0),
\end{aligned}$$

and

$$\{G_2^{h=0}(x), G_3^{h=0}(z)\} = - \sum_{i=1}^3 \frac{d\mathcal{V}}{dp_i} G_i^g.$$

The tricky parts are the second and third bracket in (B.1),

$$\begin{aligned} \{(q^3 h(p_1) g(\overline{\chi}\chi))[\varphi], G_3^{h=0}[\psi]\} &= \iint dx dz \varphi_x \psi_z g_x(\overline{\chi}\chi) \{q_x^3 h_x(p_1), G_{3,z}^g\} \\ &= \iint dx dz \varphi_x \psi_z g_x(\overline{\chi}\chi) [(\partial_z \delta(x-z)) h_x(p_1) \\ &\quad - (q^1 h - q^3 p_3 h' - q^2 p_2 U h)_x \delta(x-z)] \\ \{G_2^{h=0}[\varphi], (q^2 h(p_1) g(\overline{\chi}\chi))[\psi]\} &= \iint dx dz \varphi_x \psi_z g_z(\overline{\chi}\chi) \{G_{2,x}^g, q_z^2 h_z(p_1)\} \\ &= \iint dx dz \varphi_x \psi_z g_z(\overline{\chi}\chi) [(-\partial_x \delta(x-z)) h_z(p_1) \\ &\quad - (q^1 h - q^2 p_2 h' - q^3 p_3 U h)_x \delta(x-z)] . \end{aligned}$$

$$\begin{aligned} \Rightarrow & \{ (q^3 h(p_1) g(\overline{\chi}\chi))[\varphi], G_3^{h=0}[\psi] \} - \{ G_2^{h=0}[\varphi], (q^2 h(p_1) g(\overline{\chi}\chi))[\psi] \} \\ = & \iint dx dz \varphi_x \psi_z [\underbrace{((\partial_z \delta(x-z)) g_x(\overline{\chi}\chi) h_x(p_1) + (\partial_x \delta(x-z)) g_z(\overline{\chi}\chi) h_z(p_1))}_{=-\partial_x \delta(x-z)} \\ & - g(q^2 p_2 - q^3 p_3)(h'(p_1) - U(p_1) h(p_1)) \delta(x-z)] \\ \stackrel{int.p.p.}{=} & \iint dx dz \varphi_x \psi_z \delta(x-z) [\partial_x(hg) - g(q^2 p_2 - q^3 p_3)(h' - Uh)] \end{aligned}$$

One arrives at an expression for the graded Poisson bracket of G_2 and G_3 reading

$$\begin{aligned} \{G_2, G_3'\} &= \left[-\frac{d\mathcal{V}}{dp_i} G_i^g + \frac{i}{\sqrt{2}} f' [p_3(Q^3 \overleftrightarrow{\partial}_x Q^1) - p_2(Q^2 \overleftrightarrow{\partial}_x Q^0)] \right. \\ &\quad \left. + \partial_x(hg) - g(q^2 p_2 - q^3 p_3)(h' - Uh) \right] \delta . \end{aligned}$$

The $C^{\alpha\beta}$ -terms of the Dirac bracket are (with $\partial_x g = g' \partial_x(\overline{\chi}\chi)$, $\partial_x f = f' \partial_x p_1$ and $p_2 q^2 - p_3 q^3 - \partial_x p_1 = -G_1$)

$$- \frac{i}{\sqrt{2}} [f' + Uf] [p_3(Q^3 \overleftrightarrow{\partial}_x Q^1) - p_2(Q^2 \overleftrightarrow{\partial}_x Q^0)] - \frac{h}{f} f' g' \cdot (\overline{\chi}\chi) G_1 - h(\partial_x g) .$$

With these results one obtains (omitting the $\delta(x - x')$)

$$\begin{aligned}
\{G_2, G'_3\}^* &= -\frac{d\mathcal{V}}{dp_1}G_1 - \frac{d\mathcal{V}}{dp_2}\left(G_2^g + \frac{i}{\sqrt{2}}f(Q^3\overleftrightarrow{\partial}_x Q^1) + q^3hg\right) \\
&\quad - \frac{d\mathcal{V}}{dp_3}\left(G_3^g - \frac{i}{\sqrt{2}}f(Q^3\overleftrightarrow{\partial}_x Q^1) - q^2hg\right) \\
&\quad + \underbrace{\partial_x(hg) - h(\partial_x g) - gh'(\partial_x p_1)}_{=0} + gh'G_1 \\
&\quad - \frac{h}{f}f'g' \cdot (\overline{\chi}\chi)G_1 \\
&= -\frac{d\mathcal{V}}{dp_i}G_i + (gh' - \frac{h}{f}f'g' \cdot (\overline{\chi}\chi))G_1.
\end{aligned}$$

B.2 Dirac Brackets used in Sec. 2.2.3

$$\begin{aligned}
\{G_1, q^{1'}\}^* &= -\partial_1\delta \\
\{G_1, q^{2'}\}^* &= q^2\delta \\
\{G_1, q^{3'}\}^* &= -q^3\delta \\
\{G_2, q^{1'}\}^* &= q^3\left[\frac{\partial\mathcal{V}}{\partial p_1} - \left(h'g - \frac{f'}{f}hg' \cdot (\overline{\chi}\chi)\right)\right]\delta \\
\{G_2, q^{2'}\}^* &= -\left[\partial_1 + q^1 - q^3\frac{\partial\mathcal{V}}{\partial p_2}\right]\delta \\
\{G_2, q^{3'}\}^* &= q^3\frac{\partial\mathcal{V}}{\partial p_3}\delta \\
\{G_3, q^{1'}\}^* &= -q^2\left[\frac{\partial\mathcal{V}}{\partial p_1} - \left(h'g - \frac{f'}{f}hg' \cdot (\overline{\chi}\chi)\right)\right]\delta \\
\{G_3, q^{2'}\}^* &= -q^2\frac{\partial\mathcal{V}}{\partial p_2}\delta \\
\{G_3, q^{3'}\}^* &= -\left[\partial_1 - q^1 + q^2\frac{\partial\mathcal{V}}{\partial p_3}\right]\delta \\
\\
\{q^i G_i, (q^i G_i)'\}^* &= -(\partial_1\delta)q^i G_i \quad (\text{no summation over } i) \\
\{q^1 G_1, (q^2 G_2)'\}^* &= -q^2 q^3 \left[\frac{\partial\mathcal{V}}{\partial p_1} - \left(h'g - \frac{f'}{f}hg' \cdot (\overline{\chi}\chi)\right)\right] G_1 \delta \\
&= -\{q^1 G_1, (q^3 G_3)'\}^* = \{q^2 G_2, (q^3 G_3)'\}^*
\end{aligned}$$

B.3 Calculations from Sec. 2.3

The BRST charge Ω is fermionic, thus the left hand side of (2.103) reads

$$\{\Omega^{(0)}, \Omega^{(1)'}\}^* + \{\Omega^{(0)'}, \Omega^{(1)}\}^*.$$

Thus if both Dirac brackets do not contain any terms proportional to derivatives of the delta function this expression is just twice the first bracket which reads

$$\begin{aligned} & \{\Omega^{(0)}, \Omega^{(1)'}\}^* \\ &= \frac{1}{2} \{c^i G_i, (c^j c^k C_{jk}^l p_l^c)'\}^* \\ &= \frac{1}{2} c^i \{G_i, C_{jk}^{l'}\}^* c^{j'} c^{k'} p_l^{c'} - \frac{1}{2} c^j c^k C_{jk}^l G_l \delta. \end{aligned}$$

The Dirac bracket $\{G_i, C_{jk}^{l'}\}^*$ contains no δ' contributions because of the following argument: There are three Poisson brackets in the Dirac bracket (2.63); the bracket between the two arguments, and the brackets between the first and second argument and the second class constraints, respectively. The first one, $\{G_i, C_{jk}^{l'}\}$, does not give δ' contributions because the constraints (2.65)-(2.67) contain derivative terms $\partial_x p_i$ and $(Q^\dagger \overleftrightarrow{\partial} Q)$, but the structure functions C_{jk}^l only depend on p_i and Q^α , cf. eqs. (2.76) - (2.79). The same reasoning holds for $\{\Phi_\alpha, C_{jk}^{l'}\}$, and the brackets $\{G_i, \phi_\alpha'\}$ are given in App. B.1. Thus all fields in the term containing $\{G_i, C_{jk}^{l'}\}^*$ can be taken at the same point,

$$c^i \{G_i, C_{jk}^{l'}\}^* c^{j'} c^{k'} p_l^{c'} = c^i \{G_i, C_{jk}^l\}^* c^j c^k p_l^c \delta(x - x'),$$

and the Dirac bracket between the constraints and the structure functions can be calculated as if the fields were ordinary variables in classical mechanics (i.e. without the delta functions in eqs. (2.43) and (2.97)). Terms quadratic in the ghosts cancel, such that only the following Dirac brackets are necessary for evaluating the first term in $\{\Omega^{(0)}, \Omega^{(1)'}\}^*$.

$$\{G_2, C_{31}^3\}^* = \{G_3, C_{12}^2\}^* = 0 \quad (\text{B.2})$$

$$\{G_1, C_{23}^1\}^* = \{G_1, C_{23}^1\} = 0 \quad (\text{B.3})$$

$$\{G_1, C_{23}^2\}^* = \{G_1, C_{23}^2\} = C_{23}^2 \delta = -U(p_1) p_3 \delta \quad (\text{B.4})$$

$$\{G_1, C_{23}^3\}^* = \{G_1, C_{23}^3\} = -C_{23}^3 \delta = U(p_1) p_2 \delta. \quad (\text{B.5})$$

Thus the left hand side of eq. (2.103) reads

$$2\{\Omega^{(0)}, \Omega^{(1)'}\}^* = -c^j c^k C_{jk}^l G_l \delta + 2c^1 c^2 c^3 U(p_1) [p_3^c p_2 - p_2^c p_3] \delta. \quad (\text{B.6})$$

It cancels eq. (2.102), but produces a new contribution which in turn is cancelled by the bracket

$$\begin{aligned}
\{\Omega^{(1)}, \Omega^{(1)'}\}^* &= \frac{1}{4} \left[c^i c^j C_{ij}{}^k \{p_k^c, (c^l c^m)'\} C_{lm}{}^n p_n^{c'} + C_{ij}{}^k p_k^c \{c^i c^j, p_n^{c'}\} (c^l c^m)' C_{lm}{}^n \right. \\
&\quad \left. + c^i c^j p_k^c \underbrace{\{C_{ij}{}^k, C_{lm}{}^n\}^*}_{=\{C_{ij}{}^k, C_{lm}{}^n\}^* \delta(x-x')} (c^l c^m p_n^c)' \right] \\
&= c^i c^j c^l C_{ij}{}^k C_{lk}{}^n p_n^c \delta(x-x') + \frac{1}{4} \underbrace{c^i c^j c^l c^m}_{=0} p_k^c p_n^c \{C_{ij}{}^k, C_{lm}{}^n\}^* \delta(x-x') \\
&= -2c^1 c^2 c^3 U(p_1) [p_3^c p_2 - p_2^c p_3] \delta.
\end{aligned}$$

Here antisymmetry of the structure functions $C_{ij}{}^k = -C_{ji}{}^k$ has been used in the penultimate step, and the explicit form of the $C_{ij}{}^k$ (cf. eqs. (2.75)-(2.79)) in the ultimate one. The homological perturbation theory thus terminates at Yang-Mills level.

For calculating eqs. (B.2)-(B.5) it is helpful to know the following Poisson brackets of the second class constraints eqs. (2.53)-(2.56) with the structure functions,

$$\begin{aligned}
\{\Phi_\alpha, C_{12}'{}^2\} &= \{\Phi_\alpha, C_{13}'{}^3\} = 0 \quad \alpha = 0, 1, 2, 3 \\
\{\Phi_0, C_{23}'{}^1\} &= \left[-\frac{i}{\sqrt{2}} f(p_1) p_2 U' Q^2 + g' \left(h' - \frac{h}{f} f' \right) Q^3 - g'' \cdot (\bar{\chi} \chi) \frac{h}{f} f' Q^3 \right] \delta \\
\{\Phi_2, C_{23}'{}^1\} &= \left[-\frac{i}{\sqrt{2}} f(p_1) p_2 U' Q^0 - g' \left(h' - \frac{h}{f} f' \right) Q^1 + g'' \cdot (\bar{\chi} \chi) \frac{h}{f} f' Q^1 \right] \delta \\
\{\Phi_1, C_{23}'{}^1\} &= \left[+\frac{i}{\sqrt{2}} f(p_1) p_3 U' Q^3 + g' \left(h' - \frac{h}{f} f' \right) Q^2 - g'' \cdot (\bar{\chi} \chi) \frac{h}{f} f' Q^2 \right] \delta \\
\{\Phi_3, C_{23}'{}^1\} &= \left[+\frac{i}{\sqrt{2}} f(p_1) p_3 U' Q^1 - g' \left(h' - \frac{h}{f} f' \right) Q^0 + g'' \cdot (\bar{\chi} \chi) \frac{h}{f} f' Q^0 \right] \delta \\
\{\Phi_0, C_{23}'{}^2\} &= -\frac{i}{\sqrt{2}} f U Q^2 \delta & \{\Phi_1, C_{23}'{}^3\} &= \frac{i}{\sqrt{2}} f U Q^3 \delta \\
\{\Phi_2, C_{23}'{}^2\} &= -\frac{i}{\sqrt{2}} f U Q^0 \delta & \{\Phi_3, C_{23}'{}^3\} &= \frac{i}{\sqrt{2}} f U Q^1 \delta \\
\{\Phi_1, C_{23}'{}^2\} &= \{\Phi_1, C_{23}'{}^2\} = \{\Phi_0, C_{23}'{}^3\} = \{\Phi_2, C_{23}'{}^3\} = 0.
\end{aligned}$$

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